# Rectilinear Motion #

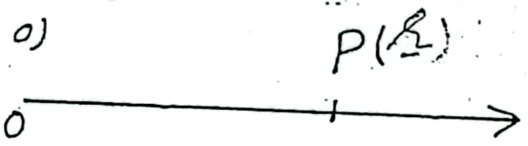
The motion of particle along a straight line is called the rectilinear motion.

## Law of Force #

The motion of a particle may involve its acceleration, Velocity, displacement and time. The Law of Force which control the motion will be a relation between acc and any one of velocity, displacement or time.

## Velocity and Acceleration #

Suppose a particle moves along x-axis and starts its motion from O. Let P(x, 0) be its position at any time t, then



$$\underline{r} = \overrightarrow{OP} = x\hat{i} + 0\hat{j} \longrightarrow ①$$

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dx}{dt}\hat{i} \longrightarrow ②$$

$$|\underline{v}| = \frac{dx}{dt}$$

$r = x\hat{i}+y\hat{j}$
$|r| = \sqrt{x^2+y^2}$

Differentiating ② w.r.t. t

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2x}{dt^2}\hat{i}$$

$$a = \text{magnitude of acc} = \frac{d^2x}{dt^2}$$

Also $a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$

**Remarks #** Acceleration can be written as

(i) $a = \frac{d^2x}{dt^2}$ (ii) $a = \frac{dv}{dt}$

(iii) $a = v\frac{dv}{dx}$

## Motion with uniform acceleration #

Consider a particle moving along a straight line with a constant acceleration (a). Let $v$ be the magnitude of velocity at any time time t. Let initially (at the start of measuring the motion)

$$t = 0 \quad x = 0 \quad v = u$$

$$Acc = a$$

$$\Rightarrow \frac{dv}{dt} = a \longrightarrow ①$$

By separating the variables

$$dv = a\,dt$$

Integrating $\int dv = a\int dt$

$$v = at + A \longrightarrow ②$$

where A is constant of integration

when $t = 0$ $v = u$

from ②

$$u = 0 + A$$

using this value of A in ②

$$v = at + u$$

$$\boxed{v = u + at} \longrightarrow ③$$

This equation gives velocity at any time.

Equation ③ can be written as

$$dx/dt = u + at$$

separating.

$$dx = (u+at)\,dt$$

$$\Rightarrow \int dx = \int (u+at)\,dt$$

$$x = ut + a\frac{t^2}{2} + B \longrightarrow ④$$

where B is constant of integration

when $t=0$ $x=0$

using in ④

$$0 = 0 + B$$

using value of B in ④

$$\boxed{x = ut + \frac{1}{2}at^2} \longrightarrow ⑤$$

This gives the distance travelled at any time t.

writing acc. as

$$v\frac{dv}{dx} = a$$

$$\int v\,dv = \int a\,dx$$

$$\frac{v^2}{2} = ax + c \longrightarrow ⑥$$

when $x=0$ $v=u$

$$\Rightarrow \frac{u^2}{2} = 0 + c \qquad \Rightarrow c = \frac{u^2}{2}$$

using value of c in ⑥

$$\frac{v^2}{2} = ax + \frac{u^2}{2}$$

$$v^2 = 2ax + u^2$$

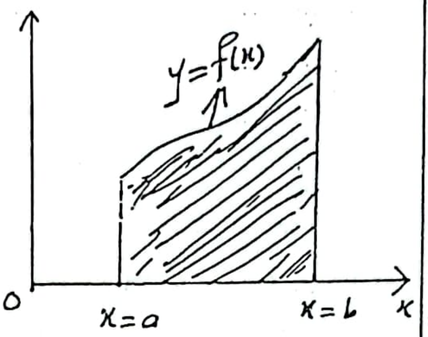
$$\boxed{2ax = v^2 - u^2} \longrightarrow ⑦$$

This gives the distance of particle in terms of initial velocity, acc and velocity at any time t

# Area under the graph of a curve #

Area under the graph of $y = f(x)$, between lines $x = a$, $x = b$ and x-axis is given by the formula

$$\text{Area} = \int_a^b y\,dx$$



# Equations for constant acceleration #

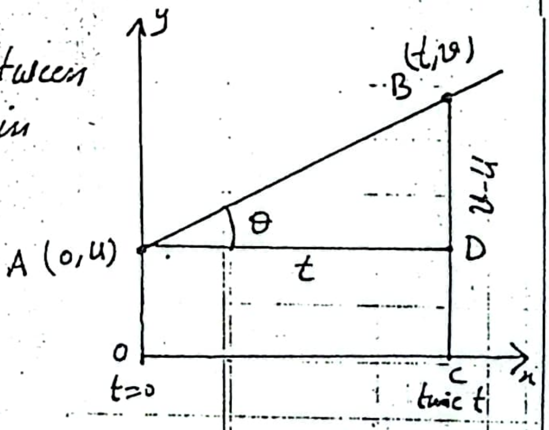
## Graphical Method

### 1) Velocity-Time graph # $V = f(t)$

If we draw a graph between velocity and time t by taking time along x-axis and velocity along y-axis, we get $(t, v)$ graph. Since by definition.

$a = \frac{dv}{dt}$ and $\frac{dv}{dt}$ is slope of the tangent at any point of $(t, v)$ graph, therefore the acc at any point is equal to the slope of tangent to t-v graph at that point.

If (a) is constant, then between $t=0$ and time t increase in velocity is $v-u$.



acc = slope of graph AB

$$a = \frac{BD}{AD}$$

$$= \frac{v-u}{t}$$

$$v-u = at \longrightarrow ①$$

Also we have

$$v = \frac{dx}{dt}$$

$$dx = v\,dt$$

Integrating

$$\int_{0}^{x} dx = \int_{t=0}^{t} v\,dt$$

$$|x|_{x=0}^{x} = \cancel{v}\int_{t=0}^{t} v\,dt$$ ∵ ~~v is constant~~

$$x = \int_{t=0}^{t} v\,dt \qquad v = f(t)$$

R.H.S of this equation is the area under the t-v graph. Thus distance covered by particle is equal to the area under the t-v graph between (0, u) and (t, v).

Now area can be calculated in two ways

$$x = \text{Area of Trapezium OABC}$$

$$= \frac{1}{2}(\text{Sum of parallel sides})(\text{Distance b/w ||al sides})$$

$$= \frac{1}{2}(u+v)t$$

$$= \frac{1}{2}(u+v)\left(\frac{v-u}{a}\right) \qquad \text{By ①}$$

$$\boxed{2ax = v^2 - u^2} \longrightarrow ②$$

Also Area can be calculated as

$$x = \text{Area of Rectangle OADC} + \text{Area } \Delta ABD$$

$$= ut + \frac{1}{2}(t)\,at = ut + \frac{1}{2}at^2$$

$$\boxed{x = ut + \frac{1}{2}at^2} \longrightarrow ④$$

**Deductions #** (i) If the particle starts with initial vel $u = 0$. Then equations of motion becomes

$$v = at \qquad x = \frac{1}{2}at^2 \qquad 2ax = v^2$$

(ii) If (a) is retardation of particle, then the equations of motion in their new form are

$$v = u - at$$
$$x = ut - \frac{1}{2}at^2$$
$$v^2 = u^2 + 2ax$$

## ertical Motion under Gravity #

If a body or eavy particle fall freely from rest (not very far from arth's surface and neglecting air resistance) vertically downwards to earth, its velocity goes on increasing with Constant rate $g = 32\, ft/sec^2$ or $9.81\, m/sec^2$ (or for Convenience $9.80\, m/sec^2$) which is known as the acc due to gravity. Then equations of motion are

$$v = gt \qquad x = \frac{1}{2}gt^2, \qquad v^2 = 2gx$$

When a heavy particle or a body is freely projected rtically upwards from a point on the earth's surface ith an initial velocity $u$ (neglecting air resistance), its locity goes on decreasing with a Constant rate known as −ve acc or retardation due to gravity. quations of motion are.

$$v = u - gt \qquad x = ut - \frac{1}{2}gt^2 \qquad v^2 = u^2 - 2gx$$

## Distance Covered in nth Seconds#

Let u be the initial velocity of the particle and (a) be the uniform acceleration. Let $x_n$, $x_{n-1}$ be the distances covered by the particle during 1st n seconds and (n−1) seconds. Then.

$$x_n = un + \frac{1}{2}an^2$$

$$x_{n-1} = u(n-1) + \frac{1}{2}a(n-1)^2$$

Distance covered in nth second
= Distance covered in n seconds − Distance covered in 1st (n−1) seconds.

$$= x_n - x_{n-1}$$

$$= un + \frac{1}{2}an^2 - \left[u(n-1) + \frac{1}{2}a(n-1)^2\right]$$

$$= u\cancel{n} + \frac{1}{2}\cancel{a}n^2 - \left[u\cancel{n} - u + \frac{1}{2}\cancel{a}n^2 + \frac{1}{2}a - an\right]$$

$$= u + an - \frac{1}{2}a$$

$$= u + \frac{1}{2}a(2n-1)$$

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## Motion With Variable Acceleration#

The acc of a moving particle may be variable and may be a function of time, velocity and distance or may be functio anyone of time, velocity, distance alone.

We shall discuss three cases.

## (a) Acceleration as a function of Time alone #

$[a = f(t)]$

The expression for acc in this case can be $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$

$$\frac{dv}{dt} = a(t)$$

$$dv = a(t)\,dt$$

Integrating

$$\int dv = \int a(t)\,dt + A$$

$$v = \varphi(t) + A$$

where $\varphi(t) = \int a(t)\,dt$

$$\frac{dx}{dt} = \varphi(t) + A$$

$$dx = \varphi(t)\,dt + A\,dt$$

Integrating

$$\int dx = \int[\varphi(t) + A]\,dt$$

$$x = \int \varphi(t)\,dt + At + B$$

where the constants A & B of integration can be determined by using the initial conditions.

## (b) # Acceleration as a function of distance alone :

$[a = f(x)]$

The expression for acc can be $v\frac{dv}{dx}$

$$v\frac{dv}{dx} = a(x)$$

$$v\,dv = a(x)\,dx$$

Integrating

$$\int v\,dv = \int a(x)\,dx + \frac{C}{2}$$

$$\frac{v^2}{2} = \psi(x) + \frac{C}{2}.$$

where c is a Constant of integration and $\psi(x) = \int a(x)\,dx$

$$v^2 = 2\psi(x) + C$$

$$v = \pm\sqrt{2\psi(x) + C}$$

$$\frac{dx}{dt} = \pm\sqrt{2\psi(x) + C}$$

$$\int dt = \pm\int \frac{dx}{\sqrt{2\psi(x)+C}} + D$$

$$t = \pm\int \frac{dx}{\sqrt{2\psi(x)+C}} + D$$

where C, D can be determined by initial Conditions

## Acceleration as a function of Velocity #

$$a = f(v)$$

$$v\cdot\frac{dv}{dx} = a(v) \quad \rightarrow ①$$

or $$\frac{dv}{dt} = a(v) \quad \rightarrow ②$$

from ① $$\int dx = \int \frac{v\,dv}{a(v)}$$

$$x = \int \frac{v\,dv}{a(v)} + E$$

where Constant of integration can be found by initial Conditions,

# Graphical Solutions of Problems #

If $y = f(x)$ is the curve in xy-plane, then $\frac{dy}{dx}$ is the slope of the tangent at any point of the curve.

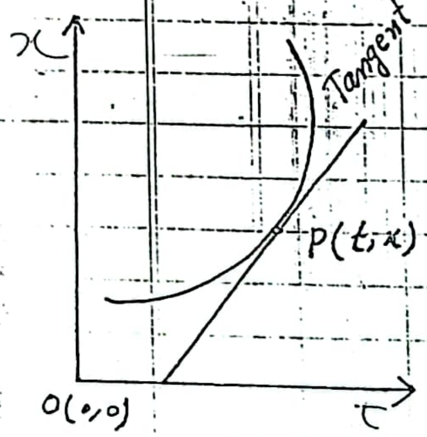
Also the integral

$$\int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} f(x)\,dx$$

gives the area bounded by the curve $y = f(x)$, x-axis and the lines $x = x_1$, $x = x_2$ parallel to y-axis

## (a) Time Space Curve #

If we draw a graph by taking time along x-axis and distance x along y-axis we get time-space curve. So the curve $x = x(t)$ is called space-time curve. Since $v = \frac{dx}{dt}$, therefore the velocity of particle at a distance x and at time t is the slope of tangent to time-space curve at point (t, x).



If Velocity of the particle is Constant, then slope of the curve remains same i.e the curve is a straight line and this can be proved mathematically as

$$v = \text{Constant} = c$$

$$\frac{dx}{dt} = c$$

$\Rightarrow \quad dx = c\,dt$

Integrating

$$\int dx = c\int dt$$

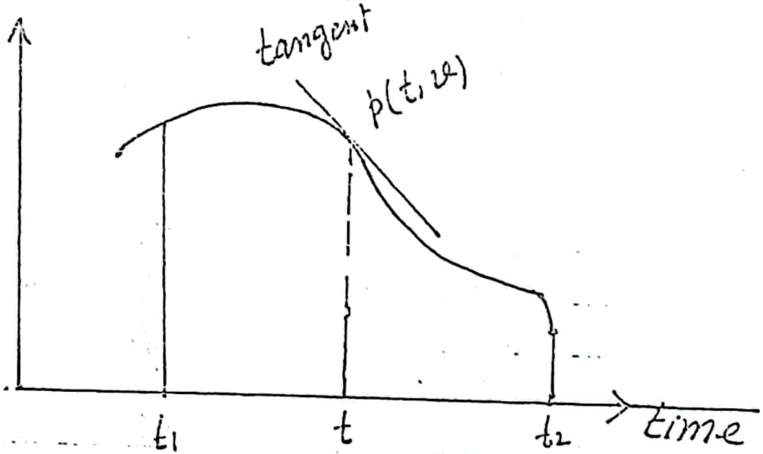
$$x = ct + A$$

which is linear equation in $x$ & $t$ with slope $= c = \frac{dx}{dt} = v =$ constant

$\Rightarrow$ when vel is constant the particle moves along a straight line. and its acc is zero

## Velocity-Time Curve #

$v = v(t)$



If we plot a graph between velocity and time by taking time along x-axis and velocity along y-axis, we get velocity-time curve.

Since acc of particle at any time is

$$a = \frac{dv}{dt}$$

Therefore slope of the tangent at any point of velocity-time curve is the acc of particle at that point.

# Distance as the area under velocity Time Curve

Since $$v = \frac{dx}{dt}$$

$$dx = v\,dt$$

Integrating $$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v\,dt$$

$$x_2 - x_1 = \int_{t}^{t_2} v\,dt$$

The R.H.S of this equation respresents the area under) bounded by the velocity time curve, t-axis and lines $t = t_1$ ; $t = t_2$ while the L.H.S gives the distance travelled by the particle in the Time interval $[t_1, t_2]$ or increase in the displacement in the time interval $[t_1, t_2]$. Therefore the area under the curve is the distance covered by the particle.

Further if the particle moves with uniform acceleration, the slope of the tangent at every point on the t-v graph will be Constant and t-v curve will be straight line which can be proved mathematically as under

$$a = \frac{dv}{dt} = \text{Constant} = k$$

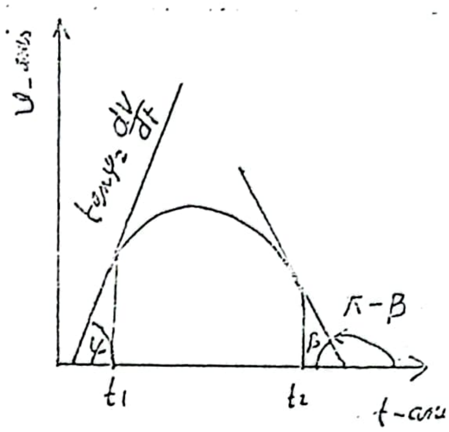
$$\int dv = k\int dt$$

$$v = kt + c$$

which is equation of line in t & v.
Thus t-v curve is straigh line.

If particle at any time $t$ has a retardation $r$, then in t-v curve the inclination of the tangent $\pi - \beta$ will be such that

$$-k = \tan(\pi - \beta)$$
$$-k = -\tan\beta$$
$$k = \tan\beta$$



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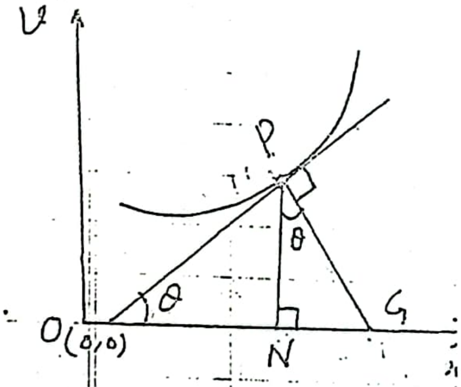
## Velocity Space Curve # $V = V(x)$

The curve $v = v(x)$ in $xv$-plane is called velocity-space curve. The slope of the curve at any point is given by, $\frac{dv}{dx}$
$\frac{dv}{dx}$



If $\theta$ is the angle of tangent at $P$ with $x$-axis, Then $\tan\theta = \frac{dv}{dx}$. Let $PG$ be the normal to the curve at $P$ and $PN$ be the perpendicular to $x$-axis. Then $\angle NPG = \theta$ and the sub-normal $NG$ is given by

$$\underline{NG} = PN\tan\theta$$
$$= v\tan\theta$$
$$= v\frac{dv}{dx} = \text{acceleration of the particle at point } P.$$

Thus the length of the subnormal at a point of the velocity space curve gives the corresponding acceleration of the particle.

Example # Find the distance travelled and the velocity attained by a particle moving in straight line, at any time t if it starts from rest at $t=0$ and is subject to an acceleration $t^2+\sin t+e^t$.

Sol # Since the particle starts from rest at any time $t=0$, therefore

at $t=0$ distance $x=0$ , $v=0$

Now

$$acc = t^2+\sin t+e^t$$

$$\frac{d^2x}{dt^2} = t^2+\sin t+e^t$$

Integrating w.r.t $t$

$$v = \frac{dx}{dt} = \frac{t^3}{3}+(-\cos t)+e^t+A \longrightarrow ①$$

At $t=0$ $v=0$

$$\Rightarrow 0 = -1+1+A$$

$$A=0$$

putting $A=0$ in ①

$$v = \frac{t^3}{3}-\cos t+e^t \longrightarrow ②$$

$$\Rightarrow \frac{dx}{dt} = \frac{t^3}{3}-\cos t+e^t$$

Integrating

$$x = \frac{t^4}{12}-\sin t+e^t+B \longrightarrow ③$$

at $t=0$ $x=0$

$$\Rightarrow 0 = 0-0+1+B$$

$$\Rightarrow B=-1$$

putting $B=-1$ in ③. Distance travelled by particle at any time $t$ is

$$x = \frac{t^4}{12}-\sin t+e^t-1$$

**Example** A particle moves in a straight line with an acceleration $kv^3$. If its initial velocity is $u$, find the velocity and the time spent when the particle has travelled a distance $x$.

**Sol#** Given that

Initial vel $= u$ at time $t=0$

$$acc = kv^3$$

Since given acc is in terms of $v$, Therefore we use expression $v\frac{dv}{dx}$ for acc and have

$$v\frac{dv}{dx} = kv^3$$

$$\int\frac{dv}{v^2} = k\int dx$$

$$-\frac{1}{v} = kx + A \longrightarrow ①$$

at $x=0$ $v = u$

$$\Rightarrow -\frac{1}{u} = A$$

putting this value in ①

$$-\frac{1}{v} = kx - \frac{1}{u}$$

$$\frac{1}{v} = \frac{1}{u} - kx = \frac{1-ukx}{u}$$

Reciprocating,

$$\boxed{v = \frac{u}{1-ukx}} \longrightarrow ②$$

Also

$$\frac{dv}{dt} = kv^3$$

$$\int \frac{dv}{v^3} = \int k\,dt$$

$$-\frac{1}{2v^2} = kt + c \longrightarrow ③$$

at $t=0$ $\quad v = u$

$$\Rightarrow \quad -\frac{1}{2u^2} = 0 + c$$

using this value of c in ③

$$-\frac{1}{2v^2} = kt - \frac{1}{2u^2}$$

$$kt = \frac{1}{2u^2} - \frac{1}{2v^2}.$$

$$t = \frac{1}{k}\left[\frac{1}{2u^2} - \frac{1}{2v^2}\right]$$

$$= \frac{1}{2ku^2}\left[1 - \frac{u^2}{v^2}\right]$$

using $v = \dfrac{u}{1-ukx}$ from ②

$$t = \frac{1}{2ku^2}\left[1 - \frac{(1-ukx)^2 . u^2}{u^2}\right]$$

$$= \frac{1}{2ku^2}\left[\cancel{1} - \cancel{1} + u^2k^2x^2 + 2ukx\right]$$

$$= \frac{1}{2ku^2} \cdot ukx\left[2 - ukx\right]$$

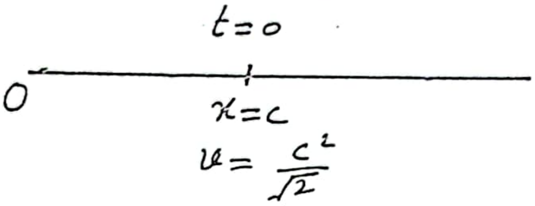
$$t = \frac{x}{2u}\left[2 - ukx\right]$$

Note # We have to find time in terms of known quantities and therefore we eliminate $v$ because $v$ has been found in terms of known quantities.

**Example #** Discuss the motion of particle moving in a straight line with an acceleration $x^3$, where $x$ is the distance of the particle from a fixed point O on the line if it starts at $t=0$ from a point $x=c$ with a velocity $\frac{c^2}{\sqrt{2}}$

**Sol #** Given that $u = \frac{c^2}{\sqrt{2}}$ at $t=0$

Also at $t=0$ $x = c$ from O



$$acc = x^3$$

$$v\frac{dv}{dx} = x^3$$

$$\int v\,dv = \int x^3\,dx$$

$$\frac{v^2}{2} = \frac{x^4}{4} + A \longrightarrow ①$$

at $x = c$ $\quad v = \frac{c^2}{\sqrt{2}}$

$$\Rightarrow \frac{\left(\frac{c^2}{\sqrt{2}}\right)^2}{2} = \frac{x^4}{4} + A$$

$$\frac{c^4}{4} = \frac{x^4}{4} + A \Rightarrow A = 0$$

using $A=0$ in ① Velocity at any distance $x$ from O is given by

$$\frac{v^2}{2} = \frac{x^4}{4}$$

$$\Rightarrow v^2 = \frac{1}{2}x^4$$

$$\Rightarrow \boxed{v = \frac{x^2}{\sqrt{2}}} \longrightarrow ②$$

$$\frac{dx}{dt} = \frac{x^2}{\sqrt{2}}$$

$$\int \frac{\sqrt{2}}{x^2}\,dx = \int dt$$

$$-\frac{\sqrt{2}}{x} = t + B \longrightarrow ③$$

at $t = 0$ $\quad x = c$

$$\Rightarrow \quad -\frac{\sqrt{2}}{c} = 0 + B$$

putting this value of B in ③ we get the time taken by the particle to cover distance $x$ from 0

$$-\frac{\sqrt{2}}{x} = -\frac{\sqrt{2}}{c} + t$$

$$t = \frac{\sqrt{2}}{c} - \frac{\sqrt{2}}{x}$$

$$\boxed{t = \sqrt{2}\left(\frac{1}{c} - \frac{1}{x}\right)}$$

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**Example #** A particle starts from rest with a constant acceleration. When its velocity acquires a certain value $v$, it moves uniformly and then its velocity starts decreasing with a constant retardation $2a$ till it comes to rest. Find the distance travelled by the particle, if the time taken from rest to rest is $t$.

**Solution #** Given that

Constant acc $= a$. $\quad$ Constant retardation $= -2a$

Starting velocity $= 0$

maximum velocity attained $= v$

time from rest to rest $= t$

Let $t_1, t_2, t_3$ be times spent during accelerated motion, uniform motion and retarded motion respectively. Then

$$\text{total time} = t_1 + t_2 + t_3$$

$$t = t_1 + t_2 + t_3 \longrightarrow ①$$

## accelerated motion#

Initial velocity $= u = 0$ acc $= a$

Time taken $= t_1 = ?$

final velocity $= v$

Let $x_1$ be the distance covered.

$$v = u + at$$

$$v = 0 + at_1$$

$$\boxed{t_1 = \frac{v}{a}} \longrightarrow \text{②}$$

Also distance covered $x_1$ is given by

$$2ax_1 = v^2 - u^2$$

$$= v^2 - 0$$

$$\boxed{x_1 = \frac{v^2}{2a}} \longrightarrow \text{③}$$

## Retarded motion#

Initial velocity $= v$

Retardation $= -2a$

Time taken $= t_3 = ?$

final velocity $= 0$

Let $x_3$ be the distance covered during motion.

$$v = u + at$$

$$0 = v - 2at_3$$

$$\boxed{t_3 = \frac{v}{2a}} \longrightarrow \text{④}$$

Distance covered is given by

$$2ax = v^2 - u^2$$

$$2(-2a)x_3 = 0 - v^2$$

$$\boxed{x_3 = \frac{v^2}{4a}} \longrightarrow \text{⑤}$$

## Uniform Motion #

When particle attains velocity V, it starts moving uniformly with this velocity.

So

$$vel = v$$

from ① $t_2 = t - t_1 - t_3$

putting $t_1 = \frac{v}{a}$ from ② & $t_3 = \frac{v}{2a}$ from ④

$$t_2 = t - \frac{v}{a} - \frac{v}{2a} = t - \frac{3v}{2a}$$

Let $x_2$ be the distance covered. Then

$$x_2 = vt_2.$$

$$\boxed{x_2 = v\left(t - \frac{3v}{2a}\right)} \longrightarrow ⑥$$

## Distance Covered #

using ③ & ⑤ & ⑥, we have the total distance covered by particle

Distance covered from rest to rest

$$= x_1 + x_2 + x_3.$$

$$= \frac{v^2}{2a} + v\left(t - \frac{3v}{2a}\right) + \frac{v^2}{4a}$$

$$= \frac{3v^2}{4a} + vt - \frac{3v^2}{2a}$$

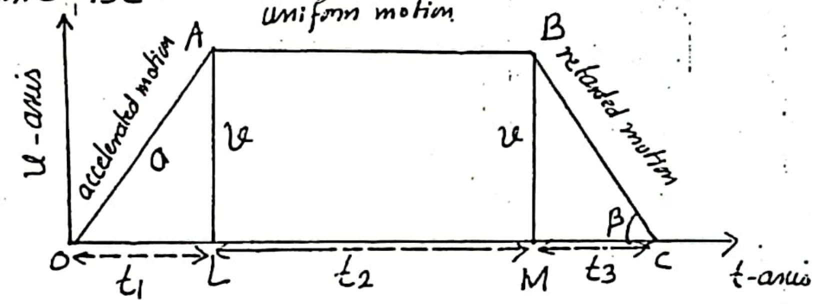
$$= vt - \frac{3v^2}{4a} = \frac{1}{2}v\left(2t - \frac{3v}{2a}\right)$$

## Graphical Method #

We draw the velocity time curve of the motion. Since particle starts its motion from rest with uniform acc (a), therefore graph of accelerated motion is straight line OA, where AL = v

During uniform motion the partide moves with uniform velocity $v$, therefore its acc $\frac{dv}{dt} = 0$ i.e slope at every point of tre-curve is zero and tre graph for this part of motion is parllel to $t$-axis and is denoted by $AB$ in fig.

Finally the partide moves with constant retardation during retarted motion and grph is st-line $BC$.



Let $t_1, t_2, t_3$ be times taken during accelerated, uniform and retarded motions respectively and $x_1, x_2, x_3$ be the distances covered during these motions.

## Accelerated Motion#

$$acc = a = \text{Slope of } OA = \frac{v}{t_1}$$

$$\Rightarrow \boxed{t_1 = \frac{v}{a}} \longrightarrow ①$$

$$\text{Distance covered} = x_1 = \text{Area of } \Delta OAL$$

$$= \frac{1}{2} OL \cdot AL$$

$$x_2 = \frac{1}{2} v t_1 \longrightarrow ②$$

## Uniform Motion#

$$AB = LM = t_2.$$

$$\text{Distance Covered} = x_2 = \text{Area of rectangle } ABML$$

$$x_2 = AL \cdot LM = vt_2.$$

$$x_2 = vt_2 \longrightarrow ③$$

## Retarded Motion#

Retardation = Slope of BC

$$-2a = \tan(\pi-\beta)$$

$$-2a = -\tan\beta$$

$$2a = \tan\beta$$

$$= \frac{v}{t_3}$$

$$t_3 = \frac{v}{2a} \longrightarrow ④$$

Distance covered = Area of Δ BMC

$$x_3 = \frac{1}{2} BM \cdot MC$$

$$x_3 = \frac{1}{2} v t_3 \longrightarrow ⑤$$

Distance covered from rest to rest

$$= x_1 + x_2 + x_3$$

$$= \frac{1}{2} v t_1 + v t_2 + \frac{1}{2} v t_3$$

$$= \frac{1}{2} v [t_1 + 2t_2 + t_3]$$

$$= \frac{1}{2} v [2t_1 - t_1 + 2t_2 + 2t_3 - t_3]$$

$$= \frac{1}{2} v [2t_1 + 2t_2 + 2t_3 - (t_1 + t_3)]$$

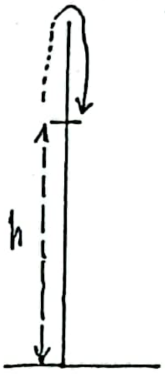
$$= \frac{1}{2} v \left[2(t_1+t_2+t_3) - \left(\frac{v}{a} + \frac{v}{2a}\right)\right]$$

$$= \frac{1}{2} v \left[2t - \frac{3v}{2a}\right] \qquad \because t_1+t_2+t_3 = t$$

**Example#** A particle is projected vertically upwards at $t=0$ with a velocity $u$, passes a point at a height $h$ at $t = t_1$ and $t = t_2$. Show that $t_1 + t_2 = \frac{2u}{g}$ and $t_1 t_2 = \frac{2h}{g}$

**Sol#** A particle projected upward will pass a point at height $h$ twice, once during upward motion and once during during. down ward motion.

The distance covered by particle in a time $t$



$$x = ut - \frac{1}{2}gt^2$$

$$h = ut - \frac{1}{2}gt^2$$

$$2h = 2ut - gt^2$$

$$\Rightarrow gt^2 - 2ut + 2h = 0 \longrightarrow ①$$

The times $t_1$ & $t_2$ when the particle is at height $h$ from the point of projection are roots of this quadratic equation

$$t_1 + t_2 = -\left(-\frac{2u}{g}\right) = \frac{2u}{g}$$

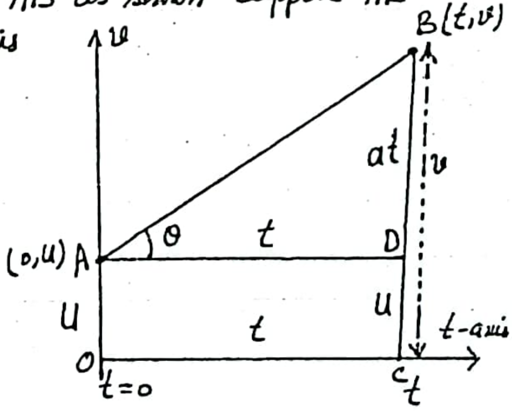
$$t_1 t_2 = \frac{2h}{g}$$

---

## Exercise #

**Q:1#** Obtain the equations of motion by graphical method

**Sol#** Suppose particle moves with uniform acc $a$, and with initial velocity $u$ at time $t=0$ and attains velocity $v$ in time $t$ after covering distance $x$. Then the graph is the straight line AB as shown. Suppose AB makes angle $\theta$ with $t$-axis



$$acc = a = \text{slope of } AB$$

$$= \tan\theta = \frac{BD}{AD}$$

$$a = \frac{BD}{t}$$

$$BD = at$$

## 1st Equation of Motion #

Velocity after time $t$ is given by

$$BC = CD + BD$$

$$v = u + at$$

## 2nd Equation of Motion #

We know that distance covered by particle is equal to the area under the $tv$-graph

Distance covered = Area under the graph

$$x = \text{Area of } \Delta ABD + \text{Area of rectangle } OADC$$

$$= \frac{1}{2} AD \cdot BD + OA \cdot OC$$

$$= \frac{1}{2} at \cdot t + u \cdot t$$

$$x = ut + \frac{1}{2}at^2$$

## 3rd Equation of Motion #

Distance covered = Area of Trapezium $OABCO$

$$= \frac{1}{2}(\text{Sum of parallel sides})(\text{Distance b/w llal sides})$$

$$x = \frac{1}{2}(OA + BC)(OC)$$

$$x = \frac{1}{2}(u+v)t$$

$$x = \frac{1}{2}(u+v)\left(\frac{v-u}{a}\right)$$

$$2ax = v^2 - u^2$$

---

**Q:2 #** A particle moving in a straight line starts from the rest and is accelerated uniform to attain a velocity of 60 miles per hour in 4 seconds. Find the acc of motion and the distance travelled by the particle in the last three seconds.

**Sol #** It is given that

Initial vel $= u = 0$ (because particle starts from rest)

Final Velocity $= v =$ 60 miles/hour

$$= \frac{\overset{88}{\cancel{60} \times 5280}}{\cancel{60} \times 60} \text{ ft/sec} \Longrightarrow$$

$$= 88 \text{ ft/sec}$$

one mile $= 1760$ yards
1 Yard $= 3$ ft
$\Rightarrow$ 1mile $= 5280$ ft

Time $= t = 4$ seconds

To find

uniform acc $= a = ?$

Distance covered in last three seconds $= ?$

**Acc #**

$$v = u + at$$

$$a = \frac{v-u}{t} = \frac{88-0}{4} = 22 \text{ ft/sec}^2$$

## Distance in Last three Seconds #

Distance Covered in 1st one second

$$= x_1 = ut + \tfrac{1}{2}at^2$$

$$= 0(1) + \tfrac{1}{2}(22)(1)^2$$

$$= 11 \text{ ft}$$

Distance Covered in four seconds

$$= x_2 = u(2) + \tfrac{1}{2}(22)(4)^2$$

$$= 0(2) + 11 \times 16 = 176 \text{ fts.}$$

Distance covered in last three seconds

$=$ Distance covered in 4 seconds $-$ distance covered in 1st one second

$$= x_2 - x_1$$

$$= 176 - 11 = 165 \text{ fts}$$

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**Q:3#** Two particles start simultaneously from a pt. O and move in a straight line; one with a velocity of 45 miles per hour and an acc of $2\ ft/sec^2$ and the other with a velocity of 90 miles/hour and a retardation of $8\ ft/sec^2$. Find the time after which the velocities of the particles are the same and the distance of O from the point where they meet again

**Sol#** Accelerated Particle#

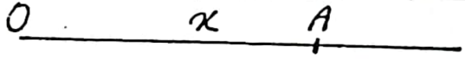
Initial velocity $= u = 45$ miles/hour

$$= \frac{45\times 5280}{60\times 60}\ ft/sec$$

$$= 66\ ft/sec \Rightarrow$$

one mile = 1760 yards
1 yard = 3 ft
⇒ 1 mile = 5280 ft

Acc $= a = 2\ ft/sec^2$



Retarded Particle#

Initial vel $= u = 90$ miles/hour

$$= \frac{\overset{3}{\cancel{90}}\times\overset{4488}{5280}}{\cancel{60}\times\underset{2}{\cancel{60}}}\ ft/sec$$

$$= 132\ ft/sec$$

Acc = retardation $= r = -8\ ft/sec^2$

Time to attain same vel#

Let the two particles attain same velocity $v$ after time $t$ seconds.

1st particle

$v = u + at$

$v = 66 + 2t \longrightarrow ①$

2nd particle

$v = u + rt$

$= 132 - 8t \longrightarrow ②$

Comparing ① & ② we have

$66 + 2t = 132 - 8t$

$10t = 66 \Rightarrow t = 6.6$ seconds

## Distance at Which the Particle Meet again #

Let $t_1$ be the Time after which the particle meet again at a distance $x$ from o

### Accelerated Particle

$$x = ut_1 + \frac{1}{2}at_1^2$$

$$= 66t_1 + \frac{1}{2}(2)t_1^2$$

$$x = 66t_1 + t_1^2 \longrightarrow ③$$

### Retarded Particle #

$$x = ut_1 + \frac{1}{2}at_1^2$$

$$= 132t_1 + \frac{1}{2}(-8)t_1^2$$

$$x = 132t_1 - 4t_1^2 \longrightarrow ④$$

from ③ & ④, we have.

$$66t_1 + t_1^2 = 132t_1 - 4t_1^2$$

$$5t_1^2 - 66t_1 = 0$$

$$t_1(5t_1 - 66) = 0$$

$$\Rightarrow t_1 = 0 \quad , \quad t_1 = \frac{66}{5} \text{ second}$$

$t_1 = 0$ corresponds to initial condition. Therefore

$$t_1 = \frac{66}{5} \text{ seconds} = 13\frac{1}{5} \text{ seconds} \qquad 5\overline{)66}\ 13,\ 65,\ 1$$

$$t_1 = 13.2 \text{ seconds}$$

putting value of $t_1$ in ③, we distance from o at which the particles meet again

$$\text{Distance} = x = 66(13.2) + (13.2)^2 = 1045.44 \text{ ft}$$

---

**Q:4 #** A particle moving along a straight line starts from rest and is accelerated uniformly till it attains a velocity $v$. The motion is then retarded and the particle comes to rest after traversing a total distance $x$. If the acc is $f$, find the retardation and total time taken by the particle from rest to rest

**Sol #**

## Accelerated Motion #

Initial vel $= u = 0$

Final velocity during motion $= v$.

Acc $= f$.

Let $x_1$ be the distance covered and $t_1$ be be time taken to attain velocity $v$

$$2ax = v^2 - u^2$$
$$2fx_1 = v^2 - 0$$
$$x_1 = \frac{v^2}{2f} \longrightarrow ①$$

Also for $t_1$

$$v = u + at$$
$$v = 0 + ft_1$$
$$\Rightarrow t_1 = \frac{v}{f} \longrightarrow ②$$

## Retarded Motion #

Initial velocity $= v$

because the particle comes to rest, therefore

Final velocity $= 0$

Let the retardation $= -r = ?$

Let $x_2$ be the distance covered and $t_2$ be the time taken during this motion.

$$2ax = v^2 - u^2$$
$$2(-r)x_2 = 0 - v^2$$
$$x_2 = \frac{v^2}{2r} \longrightarrow ③$$

For $t_2$

$$v = u + at$$
$$0 = v - rt_2$$
$$t_2 = \frac{v}{r} \longrightarrow ④$$

## Time from rest to Rest # & Retardation

total time $= t_1 + t_2$

(given) $t = \frac{v}{f} + \frac{v}{r} \longrightarrow$ ⑤

Total distance covered $= x_1 + x_2$

$$x = \frac{v^2}{2f} + \frac{v^2}{2r}$$

$$x = \frac{v}{2}\left(\frac{v}{f} + \frac{v}{r}\right)$$

$$x = \frac{v}{2}t \qquad \text{by ⑤}$$

$$\Rightarrow \boxed{t = \frac{2x}{v}}$$

Again for retardation

$$x = \frac{v^2}{2}\left(\frac{1}{f} + \frac{1}{r}\right)$$

$$\frac{2x}{v^2} = \frac{1}{f} + \frac{1}{r}$$

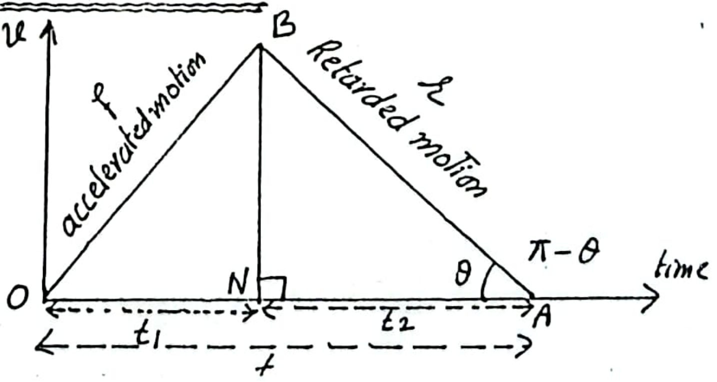
$$\Rightarrow \frac{1}{r} = \frac{2x}{v^2} - \frac{1}{f} = \frac{2xf - v^2}{v^2 f}$$

Reciprocating

$$r = \frac{v^2 f}{2xf - v^2}.$$

Q: 5 # ~~Two particles travel along a str~~

## Graphical Method #


v
B
f
accelerated motion
r
Retarded motion
π − θ
θ
time
O
N
A
t₁
t₂
t

We represent the motion in velocity-Time graph as shown. Let $t_1, t_2$ be the Times for accelerated and retarded motion.

Let $ON = t_1$ $\quad NA = t_2$

Let $r$ be the retardation.

$$\text{Acc} = \text{slope of } OB$$

$$f = \frac{v}{t_1}$$

$$\Rightarrow t_1 = \frac{v}{f} \longrightarrow ①$$

$$\text{Retardation} = -r = \text{slope of } AB$$

$$-r = \tan(\pi - \theta)$$

$$-r = -\tan\theta$$

$$r = \tan\theta = \frac{v}{t_2}$$

$$t_2 = \frac{v}{r} \longrightarrow ②$$

$$\text{Distance Covered} = \text{Area of } \Delta OAB$$

$$x = \frac{1}{2} OA \cdot BN$$

$$= \frac{1}{2}(t_1 + t_2) v$$

$$= \frac{1}{2} t v$$

$$\frac{2x}{v} = t = \text{total time taken.} \rightarrow ③$$

Also $\quad t_1 + t_2 = v\left(\frac{1}{f} + \frac{1}{r}\right)$

putting $t_1 + t_2 = \frac{2x}{v}$ from ③

$$\frac{2x}{v} = v\left(\frac{1}{f} + \frac{1}{r}\right)$$

$$\frac{2x}{v^2} = \frac{1}{f} + \frac{1}{r}$$

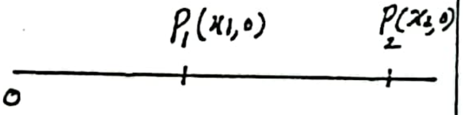
$$r = \frac{2fx - v^2}{v^2 f}$$

**Q:5 #** Two particles travel along a straight line. Both start at the same time and are accelerated uniformly at different rates. The motion is such that when a particle attains the maximum velocity $v$, its motion is retarded uniformly. The two particles come to rest simultaneously at a distance $x$ from the starting point. If the acc of 1st is $a$, and that of 2nd is $\frac{1}{2}a$, find the distance distance between the points where the two particles attain their maximum velocities.

## Sol#

### First Particle #

Suppose both particles strart from o. particle having greater acc will attain maximum vel $v$ sooner than the particle with smaller acc. Here



1st particle will attain max velocity $v$ sooner i.e at smaller distance from o.

Initial Vel $= u = 0$

Final vel $=$ max vel $= v$

acc $= a$

Let $x_1$ be the distance from o at which particle attains max velocity $v$. Then

$$2ax = v^2 - u^2$$
$$2ax_1 = v^2 - 0^2$$
$$x_1 = \frac{v^2}{2a}$$

### 2nd Particle #

Initial vel $= u = 0$

Final vel $=$ max vel $= v$

acc $= \frac{1}{2}a$

Let $x_2$ be the distance where it attains

Then

$$2ax = v^2 - u^2$$

$$2(\tfrac{1}{2}a)x_2 = v^2 - 0$$

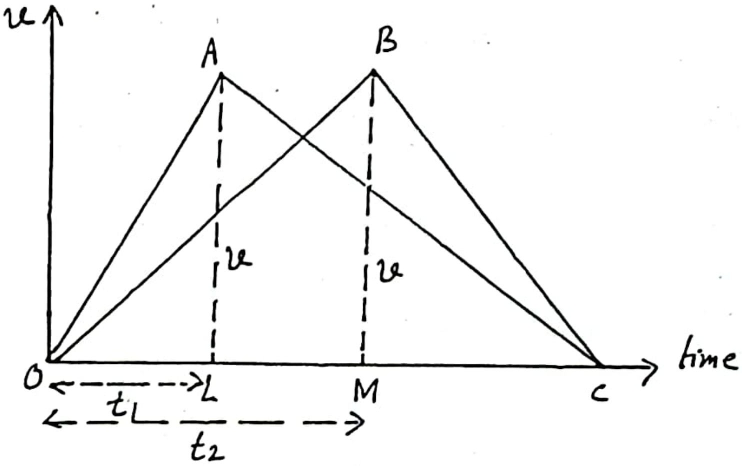
$$x_2 = \frac{v^2}{a}$$

Distance b/w points where the particles attain the max vel $v$

$$= P_1P_2 = x_2 - x_1 = \frac{v^2}{a} - \frac{v^2}{2a}$$

$$= \frac{v^2}{2a}$$

## Graphical Method #



Let $t_1$ be the taken by 1st particle and $t_2$ be the time taken by 2nd particle to attain max velocity. $v$. Then in tv-graph

Let $OL = t_1$ $\quad OM = t_2$

Let $x_1, x_2$ be distance where the particles attain the max velocity $v$.

For 1st Particle

$$acc = a = \text{slope of } OA = \frac{v}{t_1}$$

$$\Rightarrow t_1 = \frac{v}{a} \longrightarrow ①$$

$x_1 =$ distance covered when vel is $v$

$= \text{Area of } \Delta OAL$

$= \frac{1}{2} OL \cdot AL = \frac{1}{2} t_1 v$

$= \frac{1}{2}\left(\frac{v}{a}\right) v$ putting $t_1 = \frac{v}{a}$ from ①

$x_1 = \frac{v^2}{2a} \longrightarrow$ ②

## 2nd Particle #

$Acc = \frac{1}{2}a =$ slope of $OB = \frac{BM}{OM} = \frac{v}{t_2}$

$\Rightarrow \frac{1}{2}a = \frac{v}{t_2}$

$\Rightarrow t_2 = \frac{2v}{a} \longrightarrow$ ③

$x_2 =$ distance covered when vel is $v$

$= \text{Area of } \Delta OBM = \frac{1}{2} OM \cdot BM = \frac{1}{2} t_2 v$

$= \frac{1}{2}\left(\frac{2v}{a}\right) v$ Putting $t_2 = \frac{2v}{a}$ from ③

$x_2 = \frac{v^2}{a}$

Distance b/w points where the particles attain max. vel $v$

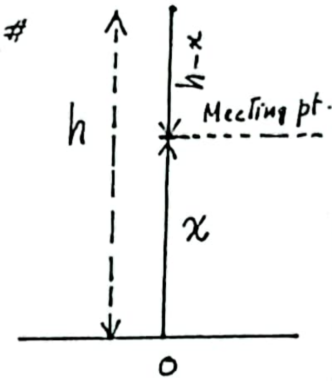
$= x_2 - x_1 = \frac{v^2}{a} - \frac{v^2}{2a} = \frac{v^2}{2a}$

---

**Q:6#** A particle is projected vertically upwards with a velocity $\sqrt{2gh}$ and another is let fall from a height $h$ at the same time. Find the height of the point where they meet each other.

**Sol#** Let O be the point of projection and the particles meet each other at height $x$ after time $t$,

## Motion of the Projected Particle #



Initial velocity = $u = \sqrt{2gh}$

Acc = $-g$

Distance covered = $x$

time taken = $t$

Equation of motion is

$$x = ut - \frac{1}{2}gt^2$$

$$x = \sqrt{2gh}\,t - \frac{1}{2}gt^2 \longrightarrow ①$$

## Motion of the falling Particle #

Intial vel = $u = 0$

acc = $g$

Distance Covered = $h-x$

time taken = $t$

Equation of motion is

$$x = ut + \frac{1}{2}gt^2$$

$$h - x = 0t + \frac{1}{2}gt^2$$

$$h - x = \frac{1}{2}gt^2 \longrightarrow ②$$

Adding ① & ②

$$h = \sqrt{2gh}\,t$$

$$\Rightarrow t = \frac{h}{\sqrt{2gh}} = \sqrt{\frac{h^2}{2gh}} = \sqrt{\frac{h}{2g}}$$

putting this value of $t$ in Equation ②

$$h - x = \frac{1}{2}g\left(\sqrt{\frac{h}{2g}}\right)^2$$

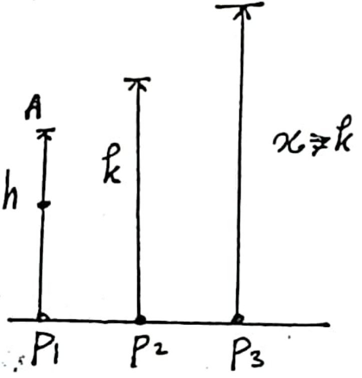
$$h - x = \frac{1}{2}g \cdot \frac{h}{2g} = \frac{h}{4}$$

$$x = h - \frac{h}{4} = \frac{3h}{4}$$ which is the required height.

**Q:7#** Two particles are projected simultaneously in the vertically upward direction with velocities $\sqrt{2gh}$ and $\sqrt{2gk}$ $(k > h)$. After a time $t$, when the two particles are still in flight, another particle is projected upwards with a vel $u$. Find the condition so that the 3rd particle may meet the 1st two particles during their upward flight.

**Sol #** In vertical upward motion, Velocity at maximum height $= 0$. Therefore to find the maximum height we must put velocity zero

Suppose particles $P_1, P_2$ are projected simultaneously with velocities $\sqrt{2gh}$ & $\sqrt{2gk}$ respectively and $P_3$ is 3rd particle which is projected after Time $t$ with velocity $u$



## For Particle $P_1$

Initial vel $= u = \sqrt{2gh}$

acc $= -g$

For max height final vel $= v = 0$

$$2ax = v^2 - u^2$$

$$2(-g)x = 0 - (\sqrt{2gh})^2$$

$$-2gx = -2gh \Rightarrow x = h$$

$\Rightarrow$ Maximum height attained $= h$.

Let $t_1$ be the time to cover height $h$. Then

$$\boxed{\cancel{x = ut - \tfrac{1}{2}gt^2}}$$

$$v = u + at$$

$$0 = \sqrt{2gh} - gt_1 \Rightarrow t_1 = \frac{\sqrt{2gh}}{g}$$

$$t_1 = \sqrt{\frac{2gh}{g^2}} = \sqrt{\frac{2h}{g}}$$

## For Particle $P_2$ #

Initial vel $= u = \sqrt{2gk}$.

acc $= -g$

Final velocity for max height $= v = 0$

$$2ax = v^2 - u^2$$

$$-2gx = 0 - (\sqrt{2gk})^2$$

$$-2gx = -2gk$$

$$x = k$$

$\Rightarrow$ Max height $= k$

Let $t_2$ be the time taken. for height $k$

$$v = u + at$$

$$0 = \sqrt{2gk} - gt_2$$

$$\Rightarrow t_2 = \sqrt{\frac{2k}{g}}$$

## Comparison between $t_1$ & $t_2$

Since $k > h$

Therefore $t_2 > t_1$

i.e $P_1$ will take shorter time to cover max heigh $h$ than that of $P_2$

Since the third particle $P_3$ is projected after time $t$ and particle $P_1$ cover height in time $t_1$, therefore $P_3$ can meet $P_1, P_2$ during their upward motion if

$$t < t_1 = \sqrt{\frac{2h}{g}}$$

so that the particle $P_1$ is still in upward motion.

Suppose after time $t$ the particle $P_1$ is at point B below its maximum height.

Time left with $P_1$ to reach height $h$ (ie to cover distance BA)

$$= t_1 - t = \sqrt{\frac{2h}{g}} - t = t'$$

## Condition on initial Velocity of $P_3$

The Third particle $P_3$ will meet the particles $P_1$ and $P_2$ in their upward motion if its initial vel $u$ is such that it covers height $> k$ in time $t'$ i.e

height covered in time $t' > k$

$$\Rightarrow \quad ut' - \frac{1}{2}gt'^2 > k$$

$$\Rightarrow \quad u\left(\sqrt{\frac{2h}{g}} - t\right) - \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t\right)^2 > k$$

Dividing by $\sqrt{\frac{2h}{g}} - t$

$$u - \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t\right) > \frac{k}{\sqrt{\frac{2h}{g}} - t}$$

$$u > \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t\right) + \frac{k}{\left(\sqrt{\frac{2h}{g}} - t\right)}$$

$$u > \frac{k}{\left(\sqrt{\frac{2h}{g}} - t\right)} + \frac{1}{2}\left(\sqrt{\frac{2g^2h}{g}} - gt\right)$$

$$u > \frac{k}{\left(\sqrt{\frac{2h}{g}} - t\right)} + \frac{1}{2}\left(\sqrt{2gh} - gt\right)$$

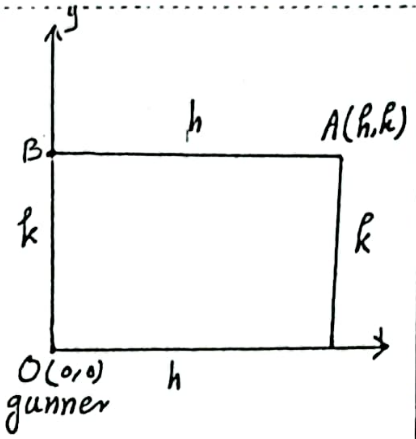
Required Condition on $u$

---

**Q:8** # A gunner detects a plane at $t=0$ approaching him with a velocity $v$, the horizontal and the vertical distances of the plane being $h$ and $k$ respectively. His gun can fire a shell vertically upwards with an initial velocity $u$. Find the time when he should fire the gun and condition on $u$ so that he may be able to hit the plane if it continues its flight in the same horizontal line.

**Sol #** Let O be the position of the gunner. Then at $t=0$ position of the plane relative to the gunner is $A(h, k)$.

The shell will hit the plane if both shell and plane reach simultaneously at point B directly above the gunner.



Let $t_1, t_2$ be the times taken by shell and plane to reach point B

## Motion of the Plane #

Since the plane is moving with constant velocity $v$, therefore

$$h = vt_2$$

$$\Rightarrow t_2 = \frac{h}{v}$$

## Motion of Shell #

Initial Vel = $u$

Distance = $k$  acc = $-g$

$$x = ut - \frac{1}{2}gt^2$$

$$k = ut_1 - \frac{1}{2}gt_1^2$$

$$gt_1^2 - 2ut_1 + k = 0$$

$$t_1 = \frac{2u \pm \sqrt{4u^2 - 8gk}}{2g}$$

$$= \frac{u \pm \sqrt{u^2 - 2gk}}{g}$$

The shell will hit the plane if the gun is fired after time

$$= t_2 - t_1 = \frac{h}{v} - \frac{u \pm \sqrt{u^2 - 2gk}}{g}$$

## Condition on $U$

The shell will actually hit the plane if time has real value i·e

if $\quad U^2 - 2gk \geq 0$

if $\quad U^2 \geq 2gk$ . Required Condition

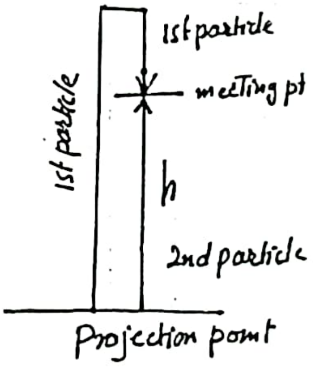
**Note** · Here time $t_2$ must be considered greater than $t_1$ because if plane takes shorter time $t_2$ to reach B than the time $t_1$ for shell · The gunner fire shell before seeing the plane at point $A(t_1 . t_2)$ (which) and the shell may not hit the plane

**Q: 9* #** A particle is projected vertically upwards. After a time $t$, another particle is sent up from the same point with the same velocity and meets the 1st at height $h$ during the downward flight of the 1st. Find the velocity of projection.

**Sol #**

### Method-I

Let $u$ be the velocity of projection of two particles and they meet after time $t_1$ starting from the time at which 2nd particle is projected



### 2nd Particle #

Distance $= h$

Time $= t_1$

vel $= u \qquad acc = -g$

Equation of motion is

$$h = ut_1 - \tfrac{1}{2}gt_1^2 \longrightarrow ①$$

### Ist Particle #

The 1st particle has spent time

$t$, when 2nd particle is projected and then after time $t_1$ it meets 2nd particle during downward flight. So 1st particle reaches height during dosward flight after time $t+t_1$. so

time = $t+t_1$
vel = $u$
Distance covered = $h$
acc = $-g$

Equation of motion

$$h = u(t+t_1) - \frac{1}{2}g(t+t_1)^2 \longrightarrow ②$$

Subtracting ② from ①

$$0 = -ut + \frac{1}{2}gt^2 + gtt_1$$

$$ut - \frac{1}{2}gt^2 = gtt_1$$

$$2ut - gt^2 = gtt_1$$

$$t_1 = \frac{2u-gt}{2g}$$

putting in ①

$$h = u\left(\frac{2u-gt}{2g}\right) - \frac{1}{2}g\left(\frac{2u-gt}{2g}\right)^2$$

$$\therefore h = \frac{2u^2-ugt}{2g} - \frac{4u^2+g^2t^2-4ugt}{8g}$$

$$= \frac{u^2}{g} - \cancel{\frac{ut}{2}} - \frac{u^2}{2g} + \frac{gt^2}{8} + \cancel{\frac{ut}{2}}$$

$$h + \frac{gt^2}{8} = \frac{u^2}{g} - \frac{u^2}{2g}$$

$$\frac{8h+gt^2}{8} = \frac{u^2}{2g}$$

$$u^2 = \frac{8hg+g^2t^2}{4}$$

$$u = \frac{\sqrt{g^2t^2+8gh}}{2}$$

# Method-II

Let $u$ be the velocity of projection and $T$ be time after which 1st particle reaches height $h$. Then

$$x = ut - \frac{1}{2}gt^2$$

$$h = uT - \frac{1}{2}gT^2$$

$$gT^2 - 2uT + 2h = 0$$

$$T = \frac{2u \pm \sqrt{u^2 - 2gh}}{g}$$

Let $t_1 = \frac{u + \sqrt{u^2 - 2gh}}{g}$ $\quad t_2 = \frac{u - \sqrt{u^2 - 2gh}}{g}$

$\Rightarrow$ $t_1$ & $t_2$ are times for 1st particle to reach height twice. Since $t_1 > t_2$, $t_2$ is time to reach during upward motion and $t_1$ is time to reach heigh $h$ during downward motion. Also since the 2nd particle has same velocity $u$, it will also take time $t_2$ reach height $h$ during upward flight.

Thus the particles will meet at height $h$ if the time after which 2nd particle is projected is such that

$$t = t_1 - t_2.$$

$$= \frac{u + \sqrt{u^2 - 2gh}}{g} - \frac{u - \sqrt{u^2 - 2gh}}{g}$$

$$t = \frac{2\sqrt{u^2 - 2gh}}{g}$$

squaring $\quad \frac{g^2t^2}{4} = u^2 - 2gh$

$$u^2 = \frac{g^2t^2}{4} + 2gh = \frac{g^2t^2 + 8gh}{4}$$

$$u = \frac{\sqrt{g^2t^2 + 8gh}}{2}.$$ Required velocity of projection.

**Q:10 #** Discuss the motion of a particle moving in a straight line if it starts from rest at $t=0$ and its acceleration is equal to (i) $t^n$ (ii) $a\cos t + b\sin t$ (iii) $-n^2x$

**Sol #** (i)

$$Acc = t^n$$
$$\frac{dV}{dt} = t^n$$
$$\int dV = \int t^n\,dt$$
$$V = \frac{t^{n+1}}{n+1} + A \longrightarrow ①$$

at $t=0$ $\quad v=0$

$$\Rightarrow \quad 0 = 0 + A$$
$$\Rightarrow \quad A = 0$$

putting in ①

$$V = \frac{t^{n+1}}{n+1}$$
$$\frac{dx}{dt} = \frac{t^{n+1}}{n+1}$$
$$\int dx = \frac{1}{n+1}\int t^{n+1}\,dt$$
$$x = \frac{1}{n+1}\,\frac{t^{n+2}}{n+2} + B \longrightarrow ②$$

at $t=0$ $\quad x=0$

$$\Rightarrow \quad B = 0$$

putting $B=0$ in ②

$$x = \frac{t^{n+2}}{(n+1)(n+2)}$$

(ii)

$$Acc = a\cos t + b\sin t$$
$$\frac{dV}{dt} = a\cos t + b\sin t$$

$$\int dV = \int (a\cos t + b\sin t)dt$$

$$v = a\sin t + b(-\cos t) + C \longrightarrow ①$$

at $t=0$ $\quad V=0$

$$0 = -b + c \Rightarrow c = b$$

putting value of $c$ in ①

$$V = a\sin t + b(-\cos t) + b$$

$$\frac{dx}{dt} = a\sin t - b\cos t + b$$

$$\int dx = \int (a\sin t - b\cos t + b)dt$$

$$x = -a\cos t - b\sin t + bt + D \longrightarrow ②$$

at $t=0$ $\quad x=0$

$$0 = -a + D \Rightarrow D = a$$

putting value of $D$ in ②

$$x = -a\cos t - b\sin t + bt + a$$

(iii) # $\quad acc = n^2 x$

$$v\frac{dV}{dx} = n^2 x$$

$$\int v\,dv = n^2\int x\,dx$$

$$\frac{v^2}{2} = n^2\frac{x^2}{2} + E \longrightarrow ①$$

at $t=0$ $\quad x=0$ $\quad v=0 \Rightarrow E=0$

using value of $E$ in ①

$$\frac{v^2}{2} = \frac{n^2x^2}{2}$$

$$v = nx$$

$$\frac{dx}{dt} = nx$$

$$\int \frac{dx}{x} = n\int \cancel{x}\,dt$$

$$\ln x = nt + F \longrightarrow ②$$

at $t = 0 \quad x = a$

$$\Rightarrow \ln a = F$$

putting in ②

$$\ln x = nt + \ln a$$

$$\ln x - \ln a = nt$$

$$\ln\left(\frac{x}{a}\right) = nt$$

$$t = \frac{1}{n}\ln\left(\frac{x}{a}\right)$$

**Q:11 #** A particle starts with velocity $u$ and moves in a straight line. If it suffers a retardation equal to the square of the velocity, find the distance travelled by the particle in a time $t$.

**Sol #** Since particle starts with a velocity $u$ and moves in a straight line and suffers retardation equal to square of velocity. Therefor if $v$ is velocity at any time $t$, then

$$acc = -v^2 (\text{retardation})$$

$$v\frac{dv}{dx} = -v^2$$

$$\int \frac{dv}{v} = -\int dx$$

$$\ln v = -x + A \longrightarrow ①$$

at $t = 0$, $x = 0$ $v = u$.

$$\Rightarrow \ln u = 0 + A \Rightarrow A = \ln u$$

putting value of $A$ in ①

$$\ln v = -x + \ln u$$

$$\ln v = -x + \ln u$$

$$x = \ln u - \ln v$$

$$= \ln\left(\frac{u}{v}\right) \longrightarrow ②$$

## Distance in terms of $t$ #

$$\frac{dv}{dt} = -v^2$$

$$\int \frac{dv}{v^2} = -\int dt$$

$$-\int \frac{dv}{v^2} = \int dt$$

$$\frac{1}{v} = t + B \longrightarrow ③$$

at $t=0$ $\quad v = u \quad B = \frac{1}{u}$

$$\frac{1}{v} = t + \frac{1}{u}$$

$$= \frac{ut+1}{u}$$

Reciprocating

$$v = \frac{u}{ut+1} \longrightarrow ④$$

putting value of $v$ from ④ in ②

Distance covered interms of known quantities is

$$x = \ln\left[\frac{u}{\frac{u}{ut+1}}\right]$$

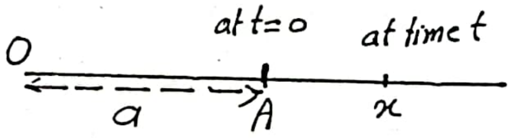
$$x = \ln(ut+1)$$

**Note** # Here we are given initial velocity $u$ & time $t$. Therefore we are to find distance covered in terms of $u$ & $t$. So we eliminate $v$ from ②

**Q:12** # Discuss the motion of particle moving in a straight line if it starts from rest at a distance 'a' from a point O and moves with an acceleration equal to $\mu$ times its distance in a time t

**Sol#** Since we are given distance from a fixed point O and acc is equal to $\mu$ times its distance at any time t, therefore we are to consider the distance of particle from O but not from the point of starting.

Let x be the distance of the particle from O at any time t.



Then according to given condition

$$acc = \mu x$$

$$v\frac{dv}{dx} = \mu x \qquad \text{(acc is function of } x\text{)}$$

$$v\,dv = \mu x\,dx$$

Integrating

$$\int v\,dv = \mu\int x\,dx$$

$$\frac{v^2}{2} = \mu\frac{x^2}{2} + A \longrightarrow ①$$

at $x = a$ $\quad v = 0$

$$\Rightarrow \quad \frac{0}{2} = \frac{\mu a^2}{2} + A$$

$$\Rightarrow \quad A = -\frac{\mu a^2}{2}$$

putting this value of A in ①

$$\frac{v^2}{2} = \frac{\mu x^2}{2} - \frac{\mu a^2}{2}$$

$$v^2 = \mu(x^2 - a^2)$$

$$v = \pm\sqrt{\mu(x^2-a^2)} \longrightarrow ②$$

But particle starts from rest i.e $v=0$, therefore its velocity will increase i.e is +ve. So we take only +ve sign in ② and.

$$v = \sqrt{\mu}\sqrt{x^2-a^2} \longrightarrow ③$$

which is the velocity of particle at any distance $x$ from $O$ after starting point

## Distance at any Time t, #

from ③

$$\frac{dx}{dt} = \sqrt{\mu}\sqrt{x^2-a^2}$$

$$\Rightarrow \int \frac{dx}{\sqrt{x^2-a^2}} = \sqrt{\mu}\int dt$$

$$\cosh^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}\,t + B \longrightarrow ③$$

at $t=0$ $x=a$

$$\Rightarrow \cosh^{-1}\left(\frac{a}{a}\right) = \sqrt{\mu}(0) + B$$

$$\cosh^{-1}(1) = B$$

$$0 = B$$

using this value of B in ③, we have

$$\cosh^{-1}(x/a) = \sqrt{\mu}\,t$$

$$\Rightarrow \frac{x}{a} = \cosh(\sqrt{\mu}\,t)$$

$$x = a\cosh(\sqrt{\mu}\,t) \longrightarrow ④$$

which is the distance of particle at any time t

Note # $\cosh^{-1}x = \ln(x+\sqrt{x^2-1}) \Rightarrow \cosh^{-1}(1) = \ln 1 = 0$

OR $\cosh x = \frac{e^x + e^{-x}}{2}$

$\Rightarrow \cosh(0) = \frac{e^0 + e^0}{2} = \frac{2}{2} = 1$

$\Rightarrow \cosh^{-1}(1) = 0$

---

**Q:13** # A particle moving in a straight line starts with a velocity $u$ and has acc $v^3$, where $v$ is the velocity of the particle at time $t$. Find the velocity and the time as functions of the distance travelled by the particle.

**Sol** # Since given acceleration is a function of $v$, so we have

$$v\frac{dv}{dx} = v^3$$

$$\int \frac{dv}{v^2} = \int dx$$

$$-\frac{1}{v} = x + A \longrightarrow ①$$

at $x = 0$ $\quad v = u$

$$\Rightarrow -\frac{1}{u} = 0 + A$$

using this value of $A$ in ①

$$-\frac{1}{v} = -\frac{1}{u} + x$$

$$= \frac{ux - 1}{u}$$

Reciprocating

$$-v = \frac{u}{ux - 1}$$

$$\text{or} \quad v = \frac{u}{1 - ux} \longrightarrow ②$$

which is required velocity in terms of $x$

## Distance & time & Time in terms of distance

from ②

$$\frac{dx}{dt} = \frac{u}{1-ux}$$

$$\int \frac{1-ux}{u}\,dx = \int dt$$

$$\frac{1}{u}\left[x - \frac{ux^2}{2}\right] = t + B \longrightarrow ③$$

At $t = 0$ $\quad x = 0$

$$\Rightarrow \quad 0 = 0 + B$$

$$B = 0$$

putting this value of B in ③

$$\frac{1}{u}\left[x - \frac{ux^2}{2}\right] = t$$

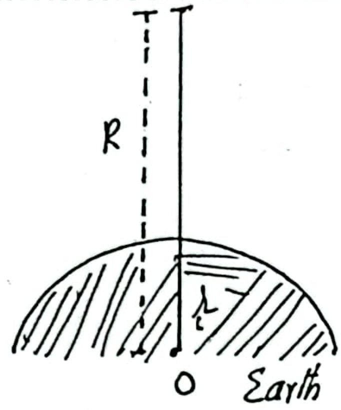
$$t = \frac{x}{2u}\left[2 - ux\right]$$

which is the required expression for time in term of distance.

**Q:14** # The acceleration of a particle falling freely under gravitational pull is equal to $\frac{k}{x^2}$, where $x$ is the distance of the particle from the centre of the earth. Find the velocity of the particle if it is let fall from an altitude $R$, on striking the surface of the earth if the radius of earth is $r$ and the air offers no resistance to motion.

**Sol** # Since the acc increases in opposite direction in which distance increase, therefore we take acc to be negative and we have

$$v\frac{dv}{dx} = -\frac{k}{x^2}$$



$$\int v\,dv = -k\int\frac{dx}{x^2}$$

$$\frac{v^2}{2} = \frac{k}{x} + A \longrightarrow ①$$

At $x = R$ $v = 0$

$$\Rightarrow \quad 0 = \frac{k}{R} + A$$

$$A = -\frac{k}{R}$$

using this value in ①

$$\frac{v^2}{2} = \frac{k}{x} - \frac{k}{R} = k\left(\frac{1}{x} - \frac{1}{R}\right)$$

$$v^2 = 2k\left(\frac{1}{x} - \frac{1}{R}\right)$$

$$v = \sqrt{2k\left(\frac{1}{x} - \frac{1}{R}\right)} \longrightarrow ②$$

(We take +ve sign because velocity increases)

Equation ② gives the velocity of particle at any distance $x$ from the surface of the earth.

When the particle strikes the earth, then $x = r$ and velocity of the particle is

Velocity on the surface of earth $= \sqrt{2k\left(\frac{1}{r} - \frac{1}{R}\right)}$

**Note** # In general altitude means height from some level. Here altitude is the distance from earth's centre

# Simple Harmonic Motion #

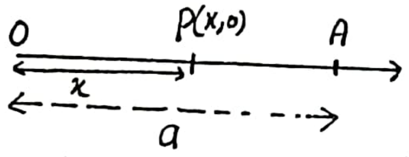
## Definition:= (Linear SHM)

The motion of a particle moving in a straight line is called linear SHM if its acceleration is always directed towards a fixed point (Mean position) in the line and is proportional to the distance of particle from that point (fixed point)

## Angular Simple Harmonic Motion #

The motion of a particle oscillating on a circular arc is called angular SHM if its angular acc is proportional to the angular displacement from a fixed line and is always directed towards that fixed line.

## Equations of Linear SHM #



Let the origin O be taken as fixed point (mean point) and particle moves along x-axis. Let $P(x,0)$ be the position of particle at any time $t$. Then according to the definition of S.H.M.

$$\text{acc} \propto -x$$

$$\Rightarrow \text{acc} = -\lambda x \quad \longrightarrow ①$$

where $\lambda$ is constant of proportionality and −ve sign indicates that acc is directed against the direction in which $x$ is increasing. Since −ve acc produces decrease in velocity, therefore ultimately the particle will come to rest $(v=0)$. Suppose the particle

Comes to rest at point A at distance (a) from O. This point A is called extreme point.

Since the acc is a function of x, therefore from ①, we have

$$v\frac{dv}{dx} = -\lambda x$$

$$\Rightarrow \int v\,dv = -\lambda\int x\,dx$$

$$\frac{v^2}{2} = -\lambda\frac{x^2}{2} + A \longrightarrow ②$$

at $x = a$ $\quad v = 0$

$$\Rightarrow 0 = -\lambda\frac{a^2}{2} + A$$

$$\Rightarrow A = \frac{\lambda a^2}{2}$$

using this value of A in ②, we get

$$\frac{v^2}{2} = -\frac{\lambda x^2}{2} + \frac{\lambda a^2}{2}$$

$$v^2 = \lambda(a^2 - x^2)$$

$$\frac{dx}{dt} = v = \pm\sqrt{\lambda}\sqrt{a^2 - x^2} \longrightarrow ③$$

This equation gives the velocity of particle at any distance x from O

Case 1 # For the direction in which x increases with time, we take $\frac{dx}{dt}$ +ve. so

$$\frac{dx}{dt} = \sqrt{\lambda}\sqrt{a^2 - x^2}$$

$$\Rightarrow \int\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\lambda}\int dt$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{a}\right) = \sqrt{\lambda}\,t + B \longrightarrow ④$$

If time is measured from A, then at $t=0 \quad x=a$ and from ④, we have

$$\sin^{-1}\left(\frac{a}{a}\right) = 0 + B$$

$$\Rightarrow \sin^{-1}(1) = B$$

$$\Rightarrow \frac{\pi}{2} = B$$

using this value in ④

$$\sin^{-1}(x/a) = \sqrt{\lambda}\,t + \frac{\pi}{2}$$

$$\Rightarrow \frac{x}{a} = \sin\left(\frac{\pi}{2} + \sqrt{\lambda}\,t\right)$$

$$\Rightarrow \frac{x}{a} = \cos\sqrt{\lambda}\,t$$

$$\boxed{\Rightarrow x = a\cos\sqrt{\lambda}\,t} \longrightarrow ⑤$$

This equation gives the distance of particle at any time t

If the time is measured from O, then at $t=0 \quad x=0$ and from ④, we have

$$\sin^{-1}(0) = \sqrt{\lambda}(0) + B$$

$$\Rightarrow 0 = 0 + B$$

$$\Rightarrow B = 0$$

using this value in ④, we get

$$\sin^{-1}(x/a) = \sqrt{\lambda}\,t$$

$$\boxed{\Rightarrow x = a\sin\sqrt{\lambda}\,t} \longrightarrow ⑥$$

Equations ⑤ & ⑥ give distance according as time measured from Ⓐ or O

**Case II #** For the direction in which distance decreases with time, we take –ve sign in ③ and so

$$\frac{dx}{dt} = -\sqrt{\lambda}\sqrt{a^2-x^2}$$

$$\Rightarrow -\int\frac{dx}{\sqrt{a^2-x^2}} = \int\sqrt{\lambda}\,dt$$

$$\Rightarrow \cos^{-1}(x/a) = \sqrt{\lambda}t + c \longrightarrow ⑦$$

If time is measured from A, then

at $t=0$ $\quad x = a$

$$\Rightarrow \cos^{-1}(1) = 0 + c$$

$$0 = c$$

using this value of c in ⑦

$$\cos^{-1}(x/a) = \sqrt{\lambda}t$$

$$\Rightarrow \boxed{x = a\cos\sqrt{\lambda}t} \longrightarrow ⑧$$

If time is measured from O, then

at $t=0$ $\quad x=0$

and from ⑦, we have

$$\cos^{-1}(0) = 0 + c$$

$$\pi/2 = c$$

using this value in ⑦

$$\cos^{-1}(x/a) = \sqrt{\lambda}t + \pi/2$$

$$\Rightarrow x = a\cos\left(\frac{\pi}{2} + \sqrt{\lambda}t\right)$$

$$\Rightarrow \boxed{x = -a\sin\sqrt{\lambda}t} \longrightarrow ⑨$$

–ve sign shows that particle is to the left of O

In numerical sense distance is given by

$$x = a \sin\sqrt{\lambda}\, t$$

Thus when time is measured from A distance is given by

$$x = a \cos\sqrt{\lambda}\, t$$

and when time is measured from o, distance is given by

$$x = a \sin\sqrt{\lambda}\, t$$

## General solution of equation of motion#

$$acc = -\lambda x$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\lambda x$$

$$\frac{d^2x}{dt^2} + \lambda x = 0$$

$$(D^2+\lambda)x = 0 \qquad \frac{d}{dt} = D$$

Auxiliary equation is

$$D^2 + \lambda = 0$$

$$D^2 = -\lambda = i^2\lambda$$

$$D = \pm i\sqrt{\lambda}$$

$$x = C_1 \sin\sqrt{\lambda}t + C_2 \cos\sqrt{\lambda}t \longrightarrow ⑩$$

Let $C_1 = a\cos\alpha \qquad C_2 = a\sin\alpha.$

⑩ become

$$x = a[\sin\sqrt{\lambda}t\cos\alpha + \cos\sqrt{\lambda}t\sin\alpha]$$

$$x = a\sin(\sqrt{\lambda}t+\alpha) \longrightarrow ⑪$$

where $\sqrt{\lambda}t+\alpha$ is called as phase at time $t$ and at $t=0$ its value is $\alpha$ which is called initial phase of motion.

Now if time is measured from A, then at $t=0 \qquad x = a$ and from ⑪

$$a = a\sin(\sqrt{\lambda}\cdot 0+\alpha)$$

$$\Rightarrow a = a\sin\alpha$$

$$\Rightarrow \sin\alpha = 1$$

$$\Rightarrow \alpha = \frac{\pi}{2}$$

putting this value of $\alpha$ in ⑪

$$x = a\sin\left(\sqrt{\lambda}\,t+\frac{\pi}{2}\right)$$

$$x = a\cos\sqrt{\lambda}\,t$$ which is as before.

If time is measured from O, then at $t=0$ $x=0$ and from ⑪

$$0 = a\sin(\sqrt{\lambda}\cdot 0+\alpha)$$

$$\Rightarrow \sin\alpha = 0$$

$$\Rightarrow \alpha = 0$$

putting this value in ⑪

$$x = a\sin\sqrt{\lambda}\,t$$ which as before.

## Amplitude of motion #

Consider

$$x = a\cos\sqrt{\lambda}\,t$$

Since $-1 \le \cos\sqrt{\lambda}\,t \le 1$

$$\Rightarrow -a \le a\cos\sqrt{\lambda}\,t \le a$$

$$\Rightarrow -a \le x \le a$$

$\Rightarrow$ particle is bounded to move from $x=-a$ to $x=a$

The distance "a" is called the amplitude of the SHM and O is called centre of motion.

## Maximum Velocity #

Velocity at any distance $x$ from o is given by

$$v = \sqrt{\lambda}\sqrt{a^2-x^2}$$

Vel is max when $x^2$ is minimum i.e when

$$x^2 = 0 \Rightarrow x = 0$$

and max velocity is obtained by putting $x=0$ in the above expression and is

$$V_{max} = \sqrt{\lambda}\, a \longrightarrow ⑫$$

## Maximum Acceleration #

Acc is numerically given by

$$acc = \lambda x$$

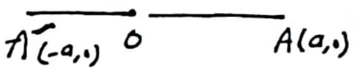
and is max when $x$ is max i.e when $x = a$

so max acc is given by

$$Max.\ acc = \lambda a \longrightarrow ⑬$$

## Oscillation or Vibration #

Particle oscillates around O upto points A(a,0) & A'(-a,0)



One complete trip from A or A' or O is called one oscillation or one vibration of particle. Following are the case for one oscillation.

With reference to point A: particles complete one oscillation as

$$A \rightarrow O \rightarrow A' \rightarrow O \rightarrow A$$

With reference to A' one oscillation is as under

$$A' \rightarrow O \rightarrow A \rightarrow O \rightarrow A'$$

with reference to point O one oscillation is

$$O \rightarrow A \rightarrow O \rightarrow A' \rightarrow O$$

## Time Period of motion #

Time required to

to Complete one oscillation is called time period of motion.

## Calculation of Time Period

expression for distance of the particle is

$$x = a\cos\sqrt{\lambda}\,t \longrightarrow ⑭$$

Velocity at any time $t$ is given by

$$v = \frac{dx}{dt} = -a\sqrt{\lambda}\sin\sqrt{\lambda}\,t \longrightarrow ⑮$$

Now the sine and Cosine are periodic functions of period $2\pi$. so we have from ⑭

$$x = a\cos\sqrt{\lambda}\,t$$

$$= a\cos(\sqrt{\lambda}\,t + 2\pi) = a\cos\sqrt{\lambda}\left(t + \frac{2\pi}{\sqrt{\lambda}}\right)$$

$$= a\cos(\sqrt{\lambda}\,t + 2\pi + 2\pi) = a\cos\lambda\left(t + \frac{2\pi}{\sqrt{\lambda}} + \frac{2\pi}{\sqrt{\lambda}}\right)$$

$$- - - - - - - - - - - - - - -$$

$$- - - - - - - - - - - - - -$$

Also from ⑮

$$v = -a\sqrt{\lambda}\sin\sqrt{\lambda}\,t$$

$$= -a\sqrt{\lambda}\sin(\sqrt{\lambda}\,t + 2\pi) = -a\sin\sqrt{\lambda}\left(t + \frac{2\pi}{\sqrt{\lambda}}\right)$$

$$= -a\sqrt{\lambda}\sin(\sqrt{\lambda}\,t + 2\pi + 2\pi) = -a\sin\sqrt{\lambda}\left(t + \frac{2\pi}{\sqrt{\lambda}} + \frac{2\pi}{\sqrt{\lambda}}\right)$$

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From above we note the position and velocity of particle are same after every time $\frac{2\pi}{\sqrt{\lambda}}$ i.e particle completes one oscillation in time $\frac{2\pi}{\sqrt{\lambda}}$. So

$$\text{Time period} = T = \frac{2\pi}{\sqrt{\lambda}}$$

### OR

If time is measured from extreme point A, the distance is given by

$$x = a\cos\sqrt{\lambda}\,t$$

When particle reaches point A′, $x = -a$ and we get

$$-a = a\cos\sqrt{\lambda}\,t$$
$$\Rightarrow \cos\sqrt{\lambda}\,t = -1$$
$$\Rightarrow \sqrt{\lambda}\,t = \pi$$
$$\Rightarrow t = \frac{\pi}{\sqrt{\lambda}}$$

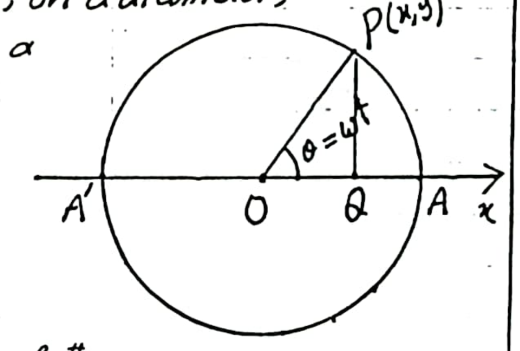
Time for back journey from A′ to A $= \frac{\pi}{\sqrt{\lambda}}$

$$\Rightarrow \text{Time period} = T = \frac{\pi}{\sqrt{\lambda}} + \frac{\pi}{\sqrt{\lambda}} = \frac{2\pi}{\sqrt{\lambda}}$$

Frequency It is the no of oscillations Completed by particle in a unit time and is given by $\nu = \frac{1}{T} = \frac{\sqrt{\lambda}}{2\pi}$

## Relation between SHM & uniform Circular motion #

Show that the projection, on a diameter, of a point moving on a circle with uniform velocity describes SHM about centre O, of the circle. Also show that the period of motion is $2\pi/\omega$ where $\omega$ is the angular speed of the point



Proof# Let $v$ be the uniform linear speed of a point moving along a circle of radius (a, . Then

$$v = a\omega \longrightarrow ①$$

Clearly $\omega = v/a$ is constant.

Time for one complete cycle $= \dfrac{\text{Distance}}{\text{vel}}$

$$= \frac{2\pi a}{v} = \frac{2\pi a}{a\omega}$$

$$= \frac{2\pi}{\omega}$$

Let Q be the projection of point on diameter AOA′ along x-axis. Then as point moves from A to A′ and back to A, its projection Q also completes one round trip about centre O of the circle.

Let P(x, y) be the position of the point on circle at any time t. Then

$$\angle AOP = \omega t$$

Co-ordinates of point Q are $(a\cos\omega t, 0)$

Distance of Q from O $= \overline{OQ} = x = a\cos\omega t \rightarrow ②$

$$\dot{x} = -a\omega \sin\omega t$$

$$\text{acc} = \ddot{x} = -a\omega^2\cos\omega t$$

$$\ddot{x} = -\omega^2(a\cos\omega t)$$

$$= -\omega^2 x \qquad \text{from① } x = a\cos\omega t$$

$$\text{acc} = -\lambda x \qquad \text{where } \lambda = \omega^2 = \text{Contat.}$$

$$\Rightarrow \text{acc} \propto -x$$

This shows that Motion of the projection Q is SHM.

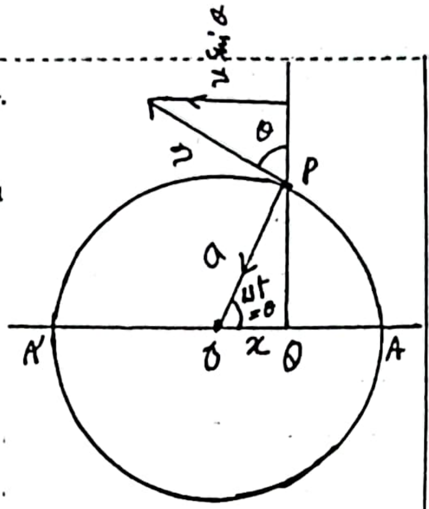
Time period $= \dfrac{2\pi}{\sqrt{\lambda}} = \dfrac{2\pi}{\sqrt{\omega^2}} = \dfrac{2\pi}{\omega}$.

(Completed)

On the Next page some addition results are drawn

Here we derive some addition results.

## Velocity of Q



The Component of Velocity $v = a\omega$ of particle P parallel to AA′ give the velocity of Q.

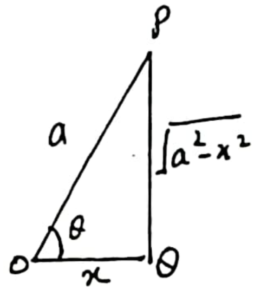
Velocity of Q = $a\omega \sin\theta$.

$= a\omega \sin\omega t$

from Δ OPQ.

$\sin\theta = \sin\omega t$

$= \frac{\sqrt{a^2-x^2}}{a}$

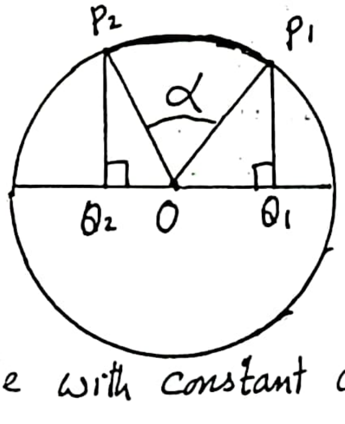


Velocity of Q = $a\omega \frac{\sqrt{a^2-x^2}}{a}$

$= \omega\sqrt{a^2-x^2}$

$= \sqrt{\lambda}\sqrt{a^2-x^2}$ $\omega^2 = \lambda \Rightarrow \omega = \sqrt{\lambda}$

The time taken by Q to traval over any section of the diameter AA′ is equal to the time taken by P to travel travel round the Corresponding arc

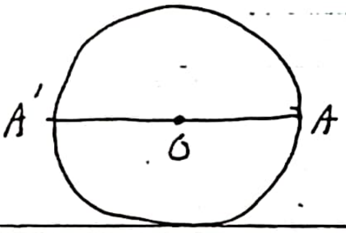


In the diagram the time t, taken to from $Q_1$ to $Q_2$ with SHM is the same sense as the time taken to move from $P_1$ to $P_2$ by particle with constant angular velocity ω.

$t = \frac{\alpha}{\omega}$, where α is measured in radius.

**Results** #★ We not the uniform motion along circle is associated with S.H.M of the projection on diameter. Hence S.H.M on line is associated with uniform motion of on circle. Here. P is projection of Q on associated circle.

★ If a particle Q performs S.H.M about O between entreme points A, A', then its projection P will have uniform circlular motion along circle with diameter AA' and centr O



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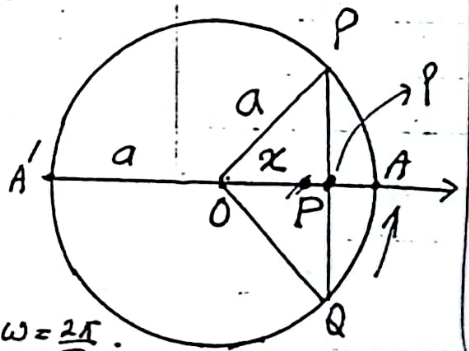
**Example** A particle performing SHM is at a distance x from the centre of motion and moving away from it. Find the time in which it will be again at the same point moving in the opposite direction. if time period of motion is T and (i) the amplitude of motion is (a) and (ii) the velocity of the particle is v.

**Sol** # (i) The motion of the particle is same as the motion of projection of a point moving along a circle of suitable radius with constant angular velocity

$$\omega = \frac{2\pi}{T}$$

Suppose the particle P initially is at distance x from the centre of motion O moving away from O. Its projection P travels with constant angular velocity $\omega = \frac{2\pi}{T}$.



Since 'a' is amplitude of motion of SHM, therefore radius of associated circle is 'a'.

Suppose projection $p'$ of $P$ at point $Q$ on the associated circle initially when $P$ is at distance $x$ from O. As $P$ moves towards, its projection on circle also moves towards A and when $P$ reaches A its projection on circle also reaches A. Now particle $P$ moves back towards initial position $P$ and its projection $P'$ on circle in the upper half circle away from A. Suppose when particle $P$ reaches its initial position $P$ at distance $x$ from O, its projection on circle is at point R.

Thus when $P$ reaches its initial position, its projection covers arc $\widehat{QR}$

Time taken by $P$ to return to initial position is same as the time taken by projection $P'$ on circle to cover arc $\widehat{QR}$ (i-e angular displacement $\angle QOR$)

Now in $\Delta\, OQR$

$$OQ = OR = a$$

$\Rightarrow$ $\Delta OQR$ is an isoceles triangle

Also $OP$ is $\perp$ar to $QR$.

$\therefore$ $OP$ bisects angle $\angle QOR$

$$\Rightarrow \angle QOR = 2\angle POR = 2\cos^{-1}(x/a)$$

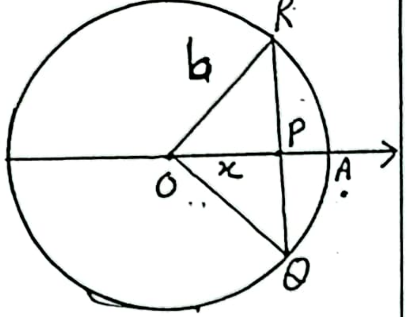
Time required to reach initial position

$= $ time taken by projection to cover an $\angle QOR$

$$= \frac{\text{angular displacement}}{\text{Constant angular vel.}} = \frac{2\cos^{-1}(x/a)}{\frac{2\pi}{T}}$$

$$= \frac{T}{2\pi}\cdot 2\cos^{-1}(x/a) = \frac{T}{\pi}\cos^{-1}(x/a)$$

(ii): Let b be the radius of the circle. Then as above



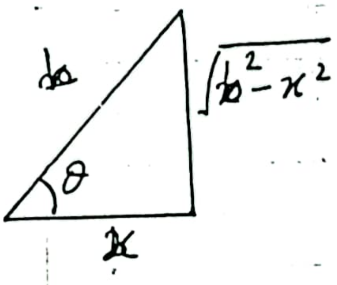
time taken by particle P to return to initial position is given by

$$t_2 = \frac{T}{\pi}\cos^{-1}\left(\frac{x}{b}\right) \longrightarrow ①$$

Let $\cos^{-1}(x/b) = \theta$

$$\Rightarrow \cos\theta = x/b$$

$$\tan\theta = \frac{\sqrt{b^2-x^2}}{x}$$



$$\Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{b^2-x^2}}{x}\right)$$

$$\Rightarrow \cos^{-1}(x/b) = \tan^{-1}\left(\frac{\sqrt{b^2-x^2}}{x}\right)$$

$$\Rightarrow t_2 = \frac{T}{\pi}\tan^{-1}\left(\frac{\sqrt{b^2-x^2}}{x}\right) \longrightarrow ②$$

But here b is unknown. We eliminate it
It is given that velocity of particle at P is v

But $v = \sqrt{\lambda}\sqrt{b^2-x^2}$

$$\Rightarrow \sqrt{b^2-x^2} = \frac{v}{\sqrt{\lambda}}$$

But given that time period $= T = \frac{2\pi}{\sqrt{\lambda}}$

$$\Rightarrow \sqrt{b^2-x^2} = \frac{v}{\frac{2\pi}{T}} = \frac{Tv}{2\pi} \qquad \Rightarrow \sqrt{\lambda} = \frac{2\pi}{T}$$

putting this value in ②

$$t_2 = \frac{T}{\pi}\tan^{-1}\left(\frac{Tv}{2\pi x}\right)$$

**Q:15#** A particle describes simple harmonic motion with frequency $N$. If the greatest velocity is $V$, find amplitude and the maximum value of the acceleration of the particle.

Also show that the velocity $v$ at a distance $x$ from the centre of motion is given by $v = 2\pi N\sqrt{a^2 - x^2}$, where 'a' is the amplitude. Ans $a = \frac{V}{2\pi N}$

Max accel $= 2\pi NV$

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**Sol#** **Given data#** frequency of motion $= N$

The greatest velocity $= V$

Distance $= x$

**Required#**

Amplitude $= ?$

Max acc $= ?$

vel at distance $x$ from centre $= ?$

First we find $\sqrt{\lambda}$

$$\text{frequency} = \frac{1}{T} = \frac{1}{\frac{2\pi}{\sqrt{\lambda}}} = \frac{\sqrt{\lambda}}{2\pi}$$

$$\Rightarrow \quad N = \frac{\sqrt{\lambda}}{2\pi}$$

$$\Rightarrow \quad \sqrt{\lambda} = 2\pi N \longrightarrow ①$$

squaring

$$\lambda = 4\pi^2 N^2 \longrightarrow ②$$

**Amplitude#**

$$\text{Max vel} = \sqrt{\lambda}\, a$$

$$V = \sqrt{\lambda}\, a$$

$$a = \frac{V}{\sqrt{\lambda}} = \frac{V}{2\pi N} \quad \text{putting } \sqrt{\lambda} \text{ from } ①$$

$$\Rightarrow \text{amplitude} = \frac{V}{2\pi N} \longrightarrow ③$$

## Maximum acceleration#

Numerical value of acc at any distance $x$ from the centre is given by

$$acc = \lambda x$$

$$\text{Maximum acc} = \lambda a$$

using values of $\lambda$ and from ② & ③

$$\text{Max acc} = 4\pi^2 N^2 \cdot \frac{V}{2\pi N}$$

$$\text{Max acc} = 2\pi N V$$ As required

## Velocity at given distance#

Velocity at distance $x$ is given by

$$v = \sqrt{\lambda}\sqrt{a^2 - x^2}$$

using value of $\sqrt{\lambda}$ from ①

$$v = 2\pi N\sqrt{a^2 - x^2}$$

where $a$ is the amplitude given by ③.

---

**l:16#** A particle describing simple harmonic motion has velocities 5ft/sec and 4ft/sec when its distances from the centre are 12ft and 13ft respectively. Find the time-period of motion.

**ol#** Given data#

Velocity of particle = 5ft/sec at distance $x = 12$ft

" " " = 4ft/sec at distance $x = 13$ft

To find:

Time period of motion = ?

formula for time period is

$$T = \frac{2\pi}{\sqrt{\lambda}} \longrightarrow ①$$

ere $\sqrt{\lambda}$ is unknown. We find it by using given data

## Calculation of $\lambda$ #

Velocity at any distance $x$ from the centre of motion is given by

$$v = \sqrt{\lambda}\sqrt{a^2 - x^2}$$

squaring

$$v^2 = \lambda(a^2 - x^2) \longrightarrow ②$$

at $x = 12ft$ $v = 5ft/se$

$$\Rightarrow (5)^2 = \lambda(a^2 - (12)^2)$$

$$\Rightarrow 25 = \lambda(a^2 - 144) \longrightarrow ③$$

at $x = 13ft$ $v = 4ft/sec$

using in ②

$$16 = \lambda(a^2 - 169) \longrightarrow ④$$

Subtracting ④ from ③

$$9 = 25\lambda \Rightarrow \lambda = \frac{9}{25}$$

$$\Rightarrow \sqrt{\lambda} = 3/5$$

using this value of $\sqrt{\lambda}$ in ①

$$\text{Time-period of motion} = \frac{2\pi}{3/5} = \frac{10\pi}{3}$$

---

**Q:17#** The maximum velocity that a particle executing simple harmonic motion of amplitude a, attains is $v$. If it is disturbed in such a way that its max. velocity becomes $nv$, find the change in magnitude of the amplitude and the time period of motion.

**Sol#** Given data # Max. velocity before disturbance $= v$

Max. Velocity after disturbance $= nv$

Velocity of particle at any distance $x$ from the centre of motion is

$$v = \sqrt{\lambda}\sqrt{a^2 - x^2}$$

$$v_{max} = \sqrt{\lambda}\, a \qquad \text{at } x = 0$$

$$\Rightarrow a = \text{amplitude} = \frac{v_{max}}{\sqrt{\lambda}}$$

But it is given $v_{max} = v$ so

$$a = \text{amplitude} = \frac{v}{\sqrt{\lambda}} \longrightarrow ①$$

Let after disturbance amplitude be $A$.

$$A = \frac{\text{max vel}}{\sqrt{\lambda}}$$

$$A = \frac{nv}{\sqrt{\lambda}} \longrightarrow ②$$

from ① $\sqrt{\lambda} = \dfrac{v}{a}$

using in ②

$$A = \frac{nv}{\frac{v}{a}} = na$$

Change in amplitude $= A - a$

$$= na - a = a(n-1)$$

$$\text{Time-period} = T = \frac{2\pi}{\sqrt{\lambda}}$$

we note that time period depends only on the constant of proportionality but not on velocity. Therefore there will be no change in time period with change in velocity and time period remains same.

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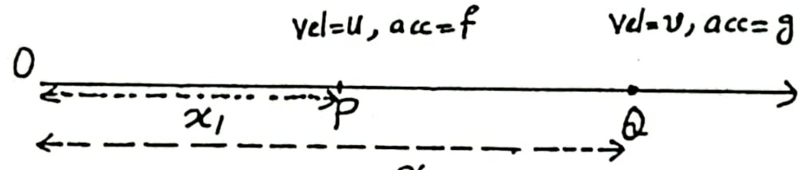
**Q:18#** If a point P moves with a velocity $v$ given by

$v^2$ (cancelled)

A point describes simple harmonic motion in such a way that its velocity and acceleration at a point P are $u$ and $f$ respectively and the corresponding quantities at another point Q are $v$ and $g$. Find the distance PQ.

**Sol#** Let O be the mean position and points P and Q be such that

$$\overline{OP} = x_1 \quad \& \quad \overline{OQ} = x_2$$



## Given Data #

At distance $x_1$ from O

$vel = u \quad \& \quad acc = f$

At distance $x_2$ from O

$vel = v \quad \& \quad acc = g$

Since we are given velocities and accelerations at points P and Q, therefore we use formulas for vel and acc.

The magnitude of acc at any distance $x$ is

$$acc = \lambda x \longrightarrow ①$$

Velocity at any distance $x$ is given by

$$vel = \sqrt{\lambda}\sqrt{a^2 - x^2}$$

squaring

$$(vel)^2 = \lambda(a^2 - x^2) \longrightarrow ②$$

**At Point P #**

$x = x_1 \qquad acc = f, \quad vel = u$

using these values in ① & ②

$$f = \lambda x_1 \longrightarrow ③$$

$$u^2 = \lambda(a^2 - x_1^2) \longrightarrow ④$$

## At Point Q #

$x = x_2$ $acc = g$ $vel = v$

Using these values in ① & ②, we get

$$g = \lambda x_2 \longrightarrow ⑤$$

$$v^2 = \lambda(a^2 - x_2^2) \longrightarrow ⑥$$

Subtracting ⑥ from ④

$$u^2 - v^2 = \lambda(x_2^2 - x_1^2)$$

$$= \lambda(x_2 - x_1)(x_2 + x_1)$$

$$= \lambda(x_1 + x_2)\overline{PQ} \quad \because \overline{PQ} = x_2 - x_1 \longrightarrow ⑦$$

Adding ③ & ⑤

$$f + g = \lambda(x_1 + x_2)$$

Using this value of $\lambda(x_1 + x_2)$ in ⑦

$$\text{Distance } PQ = \frac{u^2 - v^2}{f+g} \quad \text{as required}$$

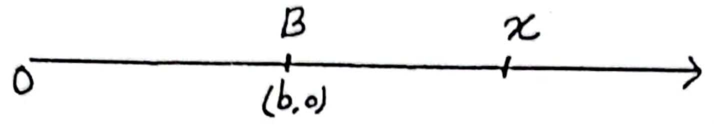
**Q: 19 #** If a point P moves with a velocity $v$ given by $v^2 = n^2(ax^2 + 2bx + c)$

Show that P executes a simple harmonic motion. Find the centre, the amplitude and the time-period of the motion. Ans $x = -b/a$, $\frac{\sqrt{b^2 - ac}}{a}$, $\frac{2\pi}{n\sqrt{a}}$

## SHM With Centre not at origin #

Before solving the question we discuss SHM whose centre is not at origion.

Let the centre of SHM of a particle be at at B(b,o)



Let x be distance of particle from O at any Time such that $x > b$.

Then distance of particle from centre. is

$$x - b$$

According to definition of SHM

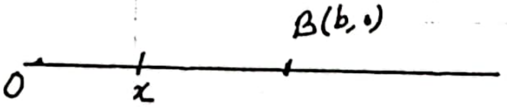
$$acc \propto -(x-b)$$

$$\Rightarrow acc = -\lambda(x-b)$$

$$\Rightarrow \frac{d^2(x-b)}{dt^2} = -\lambda(x-b)$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\lambda(x-b)$$

If at any time particle is to L.H.S of B i.e. $x < b$, Then distance of particle from B is $b-x$ and we have



$$acc \propto = -\lambda(b-x)$$

$$acc = \lambda(x-b)$$

Thus +ve sign gives the clue that particle is on left hand side of mean position.

Sol #

$$v^2 = n^2(ax^2+2bx+c)$$

Differentiating w.r.t. x

$$2v\frac{dv}{dx} = n^2(2ax+2b)$$

$$\Rightarrow v\frac{dv}{dx} = n^2(ax+b)$$
$$= n^2a\left(x+\frac{b}{a}\right)$$

Let $x + \frac{b}{a} = X$

$$\Rightarrow dx = dX$$

$$\Rightarrow v\frac{dv}{dX} = an^2X \rightarrow ①$$

Which is equation of SHM. The +ve sign on R.H.S indicates that the particle is on the left of the Centre or mean position (i.e motion is in the increasing direction of X)

Equation ① implies

$$v\frac{dv}{dX} \propto X$$

Thus particle performs S.H.M.

## Centre#

Centre of motion is given by

$$X = 0$$

$$\Rightarrow x + b/a = 0 \Rightarrow x = -b/a$$

## Amplitude# & Tim Period#

Comparing equation ①. with equation of S.H.M

$$acc = -\lambda x$$

whare $\lambda = an^2$

$$\Rightarrow \sqrt{\lambda} = \sqrt{a}\, n$$

$$\text{Time period} = T = \frac{2\pi}{\sqrt{\lambda}} = \frac{2\pi}{n\sqrt{a}}$$

Amplitude is distance on either side of centre at which velocity becomes zero.

Now vel is

$$v^2 = n^2(ax^2 + 2bx + c)$$

putting $v = 0$

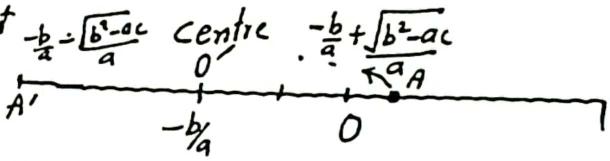
$$n^2(ax^2+2bx+c) = 0$$

$$\Rightarrow \quad ax^2+2bx+c = 0$$

$$\Rightarrow \quad x = \frac{-2b \pm \sqrt{4b^2-2ac}}{2a}$$

$$= -\frac{b}{a} \pm \frac{\sqrt{b^2-ac}}{a}$$

$\Rightarrow$ The distances of entreme points about centre $x = -b/a$ are



$-\frac{b}{a} - \frac{\sqrt{b^2-ac}}{a}$
centre
O'
$-\frac{b}{a} + \frac{\sqrt{b^2-ac}}{a}$
A
A'
$-b/a$
O

$$-b/a + \frac{\sqrt{b^2-ac}}{a} \quad \text{on R.H.S}$$

& $$-b/a - \frac{\sqrt{b^2-ac}}{a} \quad \text{on L.H.S}$$

Therefore amplitude $= \left| \frac{-b+\sqrt{b^2-ac}}{a} - (-b/a) \right|$

$$= \frac{\sqrt{b^2-ac}}{a}$$

---

## Extra Problems #

**Q:1 #** A particle moving with uniform acceleration Through 7ft and 17ft in the 1st two seconds. Find the its acceleration and its velocity after 9 seconds.

**Sol #** Given that

distance in 1st one second = 7ft

distance in 2nd second = 17ft

$\Rightarrow$ distcence covered in 2 seconds = 7+17 = 24 ft

First we find initial velocity $u$ and uniform acc $a$.

Let $u$ be the initial velocity of particle and $a$ be uniform acc.

## Motion in One Second #

distance = 7ft

time = t = 1 second

$$x = ut + \frac{1}{2}at^2$$

putting $x = 7$ $t = 1$

$$7 = u + \frac{1}{2}a$$

$$14 = 2u + a \longrightarrow ①$$

## Motion in two seconds #

Distance = 24ft

time = 2 seconds

$$x = ut + \frac{1}{2}at^2$$

putting $x = 24$ $t = 2$

$$24 = 2u + \frac{1}{2}a \cdot 4$$

$$24 = 2u + 2a \longrightarrow ②$$

Subtracting ② from ①

$$-10 = -a$$

$$\boxed{a = 10} \longrightarrow ③$$

using value of a in ①

$$14 = 2u + 10$$

$$\Rightarrow 4 = 2u$$

$$\boxed{u = 2} \longrightarrow ④$$

## Velocity after 9 seconds #

We have found

initial vel = u = 2ft/sec

acc = a = $10 ft/sec^2$

time $= t =$ 9 seconds

$v =$ Velocity after 9 seconds $= ?$

$$v = u + at$$

$$= 2 + 9\times10$$

$$v = 92 \text{ ft/sc}$$

---

**Q:2 #** If $x, y$ and $z$ are the distances covered by the particle moving with uniform acc in pth, qth and rth seconds respectively, then show that

(i) $x(q-r) + y(r-p) + z(p-q) = 0$

(ii) $p(y-z) + q(z-x) + r(x-y) = 0$

**Sol #** We know that distance covered in nth second is

Distance in nth second $= u + \frac{1}{2}a(2n-1) \longrightarrow ①$

using ① we have for distances covered in pth, qth, rth seconds.

$$x = u + \frac{a}{2}(2p-1) \longrightarrow ②$$

$$y = u + \frac{a}{2}(2q-1) \longrightarrow ③$$

$$z = u + \frac{a}{2}(2r-1) \longrightarrow ④$$

(i) Multiplying ②, ③ & ④ with $q-r, r-p, p-q$

$$x(q-r) = u(q-r) + \frac{a}{2}(2p-1)(q-r) \longrightarrow ⑤$$

$$y(r-p) = u(r-p) + \frac{a}{2}(2q-1)(r-p) \longrightarrow ⑥$$

$$z(p-q) = u(p-q) + \frac{a}{2}(2r-1)(p-q) \longrightarrow ⑦$$

Adding ⑤, ⑥ & ⑦

$$x(q-r) + y(r-p) + z(p-q) = 0$$

(ii) Subtracting ④ from ③

$$y - z = \frac{a}{2}[2q - 1 - 2r + 1]$$

$$y - z = a(q - r) \longrightarrow ⑧$$

④ − ② ⇒

$$z - x = a(r - p) \longrightarrow ⑨$$

② − ③ ⇒

$$x - y = a(p - q) \longrightarrow ⑩$$

Multiplying ⑧, ⑨, ⑩ with $p, q, r$

$$p(y - z) = pa(q - r) \longrightarrow ⑪$$
$$q(z - x) = qa(r - p) \longrightarrow ⑫$$
$$r(x - y) = ra(p - q) \longrightarrow ⑬$$

Adding ⑪, ⑫, ⑬

$$p(y - z) + q(z - x) + r(x - y) = 0$$

**Q:3** # A particle initially at rest, moves from a a fixed point in straight line so that at the end of $t$ seconds its acc is $\sin t + \frac{1}{(t+1)^2}$. Show that its distance from the fixed point at the end of $\pi$ seconds is $2\pi - \ln(\pi + 1)$

**Sol#**

$$acc = \sin t + \frac{1}{(t+1)^2}$$

$$\frac{dv}{dt} = \sin t + \frac{1}{(t+1)^2}$$

$$\int dv = \int\left[\sin t + \frac{1}{(t+1)^2}\right]dt$$

$$v = -\cos t - \frac{1}{t+1} + A \longrightarrow ①$$

Initially when $t=0$, $v=0$

$$\Rightarrow \quad 0 = -1 - 1 + A$$

$$A = 2$$

putting value of A in ①

$$v = -\cos t - \frac{1}{t+1} + 2$$

$$\Rightarrow \frac{dx}{dt} = -\cos t - \frac{1}{t+1} + 2$$

$$\int dx = \int\left(-\cos t - \frac{1}{t+1} + 2\right)dt$$

$$x = -\sin t - \ln(t+1) + 2t + B \rightarrow ②$$

Initially when $t=0$, $x=0$

$$\Rightarrow \quad 0 = 0 - 0 + 0 + B$$

$$\Rightarrow \quad B = 0$$

putting in ②

$$x = -\sin t - \ln(t+1) + 2t$$

$$x|_{t=\pi} = -\sin\pi - \ln(\pi+1) + 2\pi$$

$$= 0 + 2\pi - \ln(\pi+1)$$

$$x|_{t=\pi} = 2\pi - \ln(\pi+1) \quad \text{as required}$$

---

**Q:4 #** A point moves in a straight line starting from rest from a distance $a$, to a centre of attraction, its acceleration being $\frac{\mu}{x^3}$. Show that the time required to reach the centre is $\frac{a^2}{\sqrt{\mu}}$

**Sol #** Since the motion is attractive i.e in the decreasing direction of $x$, therefore

, Therefore

$$v\frac{dv}{dx} = -\frac{\mu}{x^3}$$

$$v\,dv = -\mu x^{-3}dx$$

$$\int v\,dv = -\mu\int x^{-3}dx$$

$$\frac{v^2}{2} = \frac{\mu}{2x^2} + \frac{A}{2} \longrightarrow ①$$

where $A$ is constant of integration

Initially, when $t=0$ $v=0$ $x=a$

$$\Rightarrow \quad 0 = \frac{\mu}{2a^2} + \frac{A}{2}.$$

$$\frac{A}{2} = -\frac{\mu}{2a^2}$$

using in ①

$$\frac{v^2}{2} = \frac{\mu}{2x^2} - \frac{\mu}{2a^2}$$

$$v^2 = \frac{\mu}{a^2x^2}(a^2-x^2)$$

$$v = \pm\frac{\sqrt{\mu}}{ax}\sqrt{a^2-x^2}$$

Since $v = \frac{dx}{dt}$ i.e $v$ depends upon $x$ & $t$.

is towards 0. So $x$ decreases with time $t$. therefore

$$\frac{dx}{dt} = -\frac{\sqrt{\mu}}{ax}\sqrt{a^2-x^2}$$

$$\int\frac{-x}{\sqrt{a^2-x^2}}dx = \int\frac{\sqrt{\mu}}{a}dt$$

$$\sqrt{a^2-x^2} = \frac{\sqrt{\mu}}{a}t + B \longrightarrow ②$$

Initially when $t=0$ $x=a$

$$\Rightarrow \quad 0 = 0 + B \Rightarrow B = 0$$

putting value of B in ②

$$\sqrt{a^2-x^2} = \frac{\sqrt{\mu}}{a}t$$

$$t = \frac{a}{\sqrt{\mu}}\sqrt{a^2-x^2}$$

The required time to reach the centre i-e when $x=0$, we have

$$t\big|_{x=0} = \frac{a^2}{\sqrt{\mu}}$$

---

**Q:5 #** A lift asends with Constant acceleration f, then with the Constant velocity and finally stops under Constant retardation f. If the total distance ascended is s and the total time occupied is t, show that the time during which the lift is ascending with Constant velocity is $\sqrt{t^2 - \frac{4s}{f}}$

## Sol# Accelerated Motion#

Let $v$ be the Constant velocity which lift attain with uniform acc f. Then

Initial velocity $= u = 0$

Distance Let $x_1$ be distance covered

$2ax = v^2 - u^2$

$2fx_1 = v^2 - 0$

$x_1 = \frac{v^2}{2f} \rightarrow$ ⓐ

acc $= f$

final velocity attained $= v$

$v = u + ft_1$

$v = 0 + ft_1$

$t_1 = \frac{v}{f} \longrightarrow$ ①

where $t_1$ is time for uniform acceleration.

## Retarded Motion#

When retardation starts lift was moving with velocity $v$. Let $t_3$ be time for lift to come to rest after retardation

Initial vel = $v$

Retardation = $-f$

Final velocity = 0

time = $t_3$ = ?

$$v = u + at$$

$$0 = v - ft_3$$

$$t_3 = \frac{v}{f} \longrightarrow ②$$

Distance

Let $x_3$ be distance covered, Then

$$2ax = v^2 - u^2$$

$$2(-f)x_3 = -v^2$$

$$x_3 = \frac{v^2}{2f} \longrightarrow (6)$$

## Uniform Motion #

uniform velocity = $v$

Let $t_2$ be time for this motion, & $x_2$ be the distance covered. Then

$$x_2 = vt_2. \longrightarrow ③$$

But total time = $t$ = $t_1 + t_2 + t_3$

$$\Rightarrow \quad t_2 = t - t_1 - t_3 \Rightarrow t_1 + t_3 = t - t_2$$

$$\Rightarrow \quad x_2 = v(t - t_1 - t_3) \longrightarrow ④$$

total distance covered = $x_1 + x_2 + x_3$

$$s = \frac{v^2}{2f} + vt_2 + \frac{v^2}{2f}$$

$$= \frac{v^2}{2f} + \frac{v^2}{2f} + vt_2.$$

$$= \frac{v}{2}\left(\frac{v}{f} + \frac{v}{f}\right) + vt_2$$

$$= \frac{v}{2}(t_1 + t_2) + vt_2$$

$$s = \frac{v}{2}(t - t_2) + vt_2$$

$$= \frac{v}{2}t - \frac{v}{2}t_2 + vt_2$$

$$= \frac{v}{2}t + \frac{v}{2}t_2$$

$$= \frac{v}{2}(t + t_2) \longrightarrow ⑤$$

Also

$$t_1 + t_3 = \frac{v}{f} + \frac{v}{f}$$

$$t - t_2 = \frac{2v}{f}$$

$$2v = (t - t_2) f$$

$$v = (t - t_2)\frac{f}{2}.$$

putting this value of $v$ in ⑤

$$s = (t - t_2)\frac{f}{2} \cdot \frac{(t + t_2)}{2}$$

$$\Rightarrow 4s = f(t^2 - t_2^2)$$

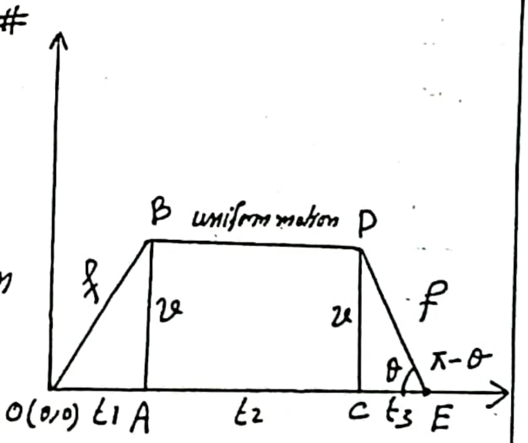
$$\frac{4s}{f} = t^2 - t_2^2$$

$$t_2^2 = t^2 - \frac{4s}{f}$$

$$t_2 = \sqrt{t^2 - \frac{4s}{f}}$$ as required

## Graphical Method #

Consider Velocity-Time graph of the motion. part OB corresponds to uniform acc, part BD Corresponds to uniform motion and Part DC corresponds to uniform retardation.



Let $v$ be the constant velocity and $t_1, t_2, t_3$ be times taken during accelerated, uniform and retarded motions respectively.

## Accelerated Motion #

$$acc = \text{slope of } OB = \frac{v}{t_1}$$

$$f = \frac{v}{t_1}$$

$$\Rightarrow t_1 = \frac{v}{f} \longrightarrow ①$$

## Retarded Motion #

$$\text{Retardation} = \text{Slope of } DE$$
$$-f = \tan(\pi-\theta)$$
$$= -\tan\theta$$
$$f = \tan\theta$$
$$= \frac{v}{t_3}$$
$$t_3 = \frac{v}{f} \rightarrow ②$$

Now

$$t = t_1+t_2+t_3$$
$$\Rightarrow t-t_2 = t_1+t_3$$
$$= \frac{v}{f}+\frac{v}{f}$$ using values of $t_1, t_3$ from ① & ②
$$v = \frac{(t-t_2)f}{2} \rightarrow ③$$

total distance = Area under the graph
= Area of $\Delta$ OAB + Area of rectangle ABDC + Area of $\Delta$ DEC.

$$s = \frac{1}{2}vt_1 + vt_2 + \frac{1}{2}vt_3$$
$$= \frac{1}{2}v[t_1+2t_2+t_3]$$
$$= \frac{1}{2}v[t_1+t_3+2t_2]$$
$$= \frac{1}{2}v[t-t_2+2t_2]$$ putting $t_1+t_3 = t-t_2$
$$= \frac{1}{2}v[t+t_2]$$
$$\Rightarrow 2s = v(t+t_2)$$

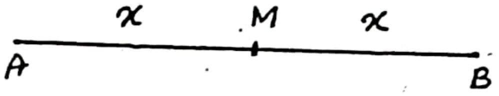
putting $v = \frac{(t-t_2)f}{2}$ from ③

$$2s = \frac{f}{2}(t+t_2)(t-t_2)$$
$$4s = f(t^2-t_2^2)$$
$$\Rightarrow t_2 = \sqrt{t^2-\frac{4s}{f}}$$ as required.

**Q:6** # Two particles P & Q move along a straight line AB with acceleration a & a'. They are simultaneously at A and have velocities u, u' respectively. If both have the same velocity at a middle point of AB, show that

$$AB = \frac{u'^2 - u^2}{a - a'}$$

**Sol** # Let $v$ be the velocity of both particle at middle point M of $AB = 2x$



$\therefore \; AM = x$

For particle P

Initial vel $= u$

acc $= a$

Velocity at $M = v$

Equation of motion is

$$2ax = v^2 - u^2 \longrightarrow ①$$

For Particle Q

Initial vel $= u'$

acc $= a'$

Velocity at $M = v$

Distance $= x$

Equation of motion is

$$2a'x = v^2 - u'^2 \longrightarrow ②$$

Subtracting ② from ①

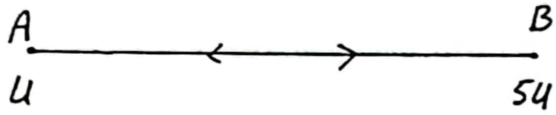
$$2x(a - a') = u'^2 - u^2$$

$$2x = \frac{u'^2 - u^2}{a - a'} =$$

$$AB = \frac{u'^2 - u^2}{a - a'}$$

**Q:7 #** A body starts with velocity $u$ and moves in a straight line with constant acc 'a'. When the velocity has become $5u$, the acc is reversed, its magnitude being unaltered. Find the velocity with which the body reaches the starting point.

**Sol #**



Let A be the starting point and B be the point where body attains its velocity $5u$.
Let $x$ be the distance between A & B

## For Direct Motion #

Initial velocity $= u$
acc $= a$
Velocity at B $= 5u$
Distance $= x = ?$

$$2ax = v^2 - u^2$$
$$2ax = (5u)^2 - u^2$$
$$2ax = 25u^2 - u^2 = 24u^2$$
$$x = \frac{24u^2}{2a} = \frac{12u^2}{a}$$

## Reversed Motion #

When acc is reversed at point B, magnitude is same but direction is towards A. Velocity of body at that time is $5u$.

So

Initial velocity $= 5u$
acc $= a$
distance $= \frac{12u^2}{a}$
final velocity at A $= v = ?$

$$v^2 - u^2 = 2ax$$

$$v^2 - (5u)^2 = 2a\left(\frac{12u^2}{a}\right)$$

$$= 24u^2$$

$$v^2 = 49u^2$$

$$v = \pm 7u$$

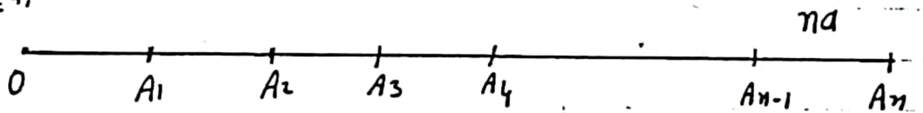
Since the initial velocity is $5u$ and it is subject to retardation.

Therefore final velocity at A can not be $7u$ but it should be $-7u$

---

**Q:8 #** A particle starts from rest with an acceleration (a), at the end of time $nt$ find its velocity and find the total distance described if its acc is $a, 2a, 3a, 4a$ ..... for time intervals $0$ to $t$, $t$ to $2t$, $2t$ to $3t$, ......

**Sol #**



Let at the end of time $t$, the particle is at $A_1$, at the end of $2t$ it is at $A_2$ and so on at the end of $(n-1)t$ it is at point $A_{n-1}$.

Let $v_1, v_2, v_3, v_4$ ------- $v_n$ be the velocities at $A_1, A_2, A_3$ ------- $A_n$ respectively

Motion from $o$ to $A_1$

Initial vel $= u = 0$ acc $= a$

time $= t$

velocity at $A_1 = v_1 = ?$

$$v = u + at$$

$$v_1 = 0 + at$$

$$\boxed{v_1 = at} \longrightarrow ①$$

## Motion from $A_1$ to $A_2$

Initial velocity $= v_1 = at$

acc $= 2a$

time $= t$ seconds

Velocity at $A_2 = v_2 = ?$

$v = u + at$

$v_2 = at + 2at = 3at = (1+2)at$

$\boxed{v_2 = 3at} \longrightarrow ②$

## Motion from $A_2$ to $A_3$

Initial velocity $= v_2 = 3at$

acc $= 3a$

time $= t$ secs

velocity at $A_3 = v_3 = ?$

$v = u + at$

$v_3 = 3at + 3at = (1+2+3)at$

$v_3 = (1+2+3)at \longrightarrow ③$

and so on generalising

Velocity at $A_n = (1+2+3+4+ \cdots + n)at$

$= \frac{n(n+1)}{2} at$

$\left\{ \text{Sum of 1st } n \text{ integers ie } 1+2+3+4+\cdots+n = \frac{n(n+1)}{2} \right\}$

Now the mean velocities in the intervals are

$\frac{v_1+0}{2}, \frac{v_1+v_2}{2}, \frac{v_2+v_3}{2}, \cdots \cdots \frac{v_{n-1}+v_n}{2}$

i.e $\frac{at}{2}, 2at, \frac{9}{2}at \cdots \cdots \frac{n^2}{2}at$

The distances travelled in the intervals are

$\frac{at^2}{2}, \frac{1}{2}(4at^2), \frac{9}{2}at^2 \cdots \cdots, \frac{n^2}{2}at^2$

Sum of the distances

$$= \frac{at^2}{2} + \frac{1}{2}4at^2 + \frac{1}{2}9at^2 + \dots\dots + \frac{1}{2}n^2at^2$$

$$= \frac{at^2}{2}[1^2 + 2^2 + 3^2 + \dots\dots + n^2]$$

$$= \frac{at^2}{2} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)(2n+1)at^2}{12}$$

Sum of the squares of 1st n natural numbers is

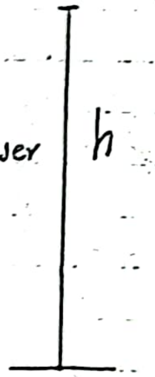
$$1^2+2^2+3^2+\dots\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

**Q:9** # A stone is dropped from the top of a tower. During the last second of its motion it covers a distance of 44.1 meters. Find the height of the tower.

**Sol** # $g = 9.8\ m/sec^2$

$u = 0$

Let h be the height of tower and the stone takes n seconds to reach the ground.



Distance covered in nth second

$$= u + \frac{1}{2}a(2n-1)$$

$$44.1 = 0 + \frac{1}{2}g(2n-1)$$

$$88.2 = 9.8(2n-1)$$

$$2n-1 = \frac{88.2}{9.8} = 9$$

$$n = 5 \text{ seconds}$$

Height of tower # Initial velocity $= u = 0$

acc $= g = 9.8\ m/sec^2$

$$\text{time} = t = 5 \text{ seconds}$$

$$\text{Distance} = x = ut + \tfrac{1}{2}at^2$$

$$h = 0 + \tfrac{1}{2}g(5)^2$$

$$= \tfrac{1}{2}(9.8)(25)$$

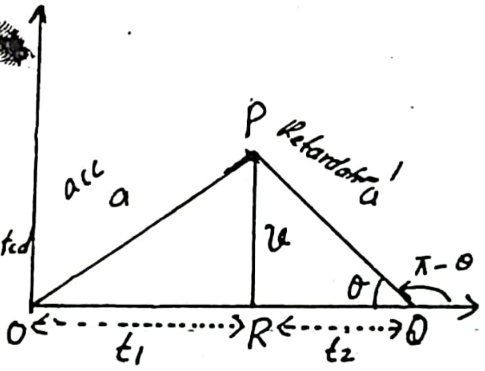
$$= (4.9)(25) = 122.5 \text{ meters.}$$

---

**Q:10 #** A car starting from a milestone stops at the next milestone after 4 minutes. If the motion is ist of uniform acceleration a and then of uniform retardation $a'$; prove that $\frac{1}{a} + \frac{1}{a'} = 8$ where the acc a is measured in kilometers per minute.

**Sol #** Let $v$ be the maximum velocity attained during accelerated motion and $t_1, t_2$ be time durations for accelerated and retarded motions.

Let OPQ be the velocity time graph as shown.



Let $OR = t_1$ $\quad RQ = t_2$ $\quad PR = v$

Portion OP of graph corresponds to accelerated motion and portion PQ is for retarded motion.

## Accelerated motion #

$$\text{acc} = \text{Slope of } OP$$

$$a = \frac{v}{t_1}$$

$$t_1 = \frac{v}{a} \longrightarrow ①$$

## Retarded motion #

$$\text{retardation} = -a' = \text{Slope of } PQ$$

$$-a' = \tan(\pi-\theta) = -\tan\theta$$

$$a' = \tan\theta = \frac{v}{t_2}$$

$$t_2 = \frac{v}{a'} \longrightarrow ②$$

Since body moves frome a mile stone to a mile stone, therefore it covers distance one kinometes in 4 minutes.

total distance = Area under the graph

$$1 = \frac{1}{2}vt_1 + \frac{1}{2}vt_2$$

$$1 = \frac{v}{2}(t_1+t_2)$$

$$= \frac{v}{2}(4) \qquad \because t_1+t_2 = 4$$

$$\Rightarrow v = \frac{1}{2}$$

Thus

$$t_1+t_2 = \frac{v}{a} + \frac{v}{a'}$$

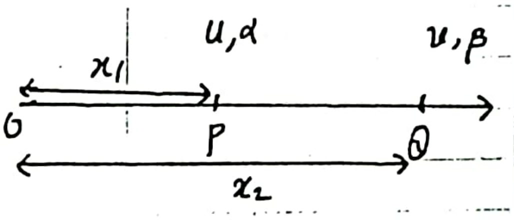
$$4 = \frac{1}{2a} + \frac{1}{2a'} \qquad \text{putting } v = \tfrac{1}{2}$$

$$8 = \frac{1}{a} + \frac{1}{a'} \quad \text{as required.}$$

---

**Q:11 #** A point describes simple harmonic motion such that in two of its positions the velocities are $u$ & $v$ and the corresponding accelerations are $\alpha, \beta$. Show that the distance between the positions is $\frac{-v^2+u^2}{\alpha+\beta}$ and find the amplitude of motion & time period.

**Sol #** Let $a$ be the amplitude of motion and $O$ be the mean position. Let $P, Q$ be such that



$$\overline{OP} = x_1 \qquad OQ = x_2$$

At point P

$acc = \alpha \quad vel = u$

At point Q

$acc = \beta \quad vel = v$

Acc and velocity of particle at any distance $x$ are given by

$$acc = \lambda x \longrightarrow ①$$

$$vel = \sqrt{\lambda}\sqrt{a^2 - x^2}$$

$$(vel)^2 = \lambda(a^2 - x^2) \longrightarrow ②$$

## At Point P#

$acc = \alpha \quad vel = u$

$vel = v \quad x = x_1$

using in ① & ②

$$\alpha = \lambda x_1 \longrightarrow ③$$

$$u^2 = \lambda(a^2 - x_1^2) \longrightarrow ④$$

## At Point Q#

$acc = \beta \quad vel = v$

$x = x_2$

using in ① & ②

$$\beta = \lambda x_2 \longrightarrow ⑤$$

$$v^2 = \lambda(a^2 - x_2^2) \longrightarrow ⑥$$

Subtracting ⑥ from ④

$$u^2 - v^2 = \lambda(x_2^2 - x_1^2)$$

$$= \lambda(x_2 + x_1)(x_2 - x_1)$$

$$= \lambda(x_2 + x_1)\overline{PQ} \longrightarrow ⑦$$

Adding ③ & ⑤

$$\alpha + \beta = \lambda(x_1 + x_2)$$

using in ⑦

$$u^2-v^2=(\alpha+\beta)\overline{PQ}$$

$$x_2-x_1=\overline{PQ}=\frac{u^2-v^2}{\alpha+\beta} \rightarrow ⑧$$

## Time Period #

⑤ − ③

$$\beta-\alpha=\lambda(x_2-x_1)$$

putting $x_2-x_1=\frac{u^2-v^2}{\alpha+\beta}$ from ⑧

$$\beta-\alpha=\lambda\left(\frac{u^2-v^2}{\alpha+\beta}\right)$$

$$\lambda=\frac{\beta^2-\alpha^2}{u^2-v^2} \longrightarrow ⑨$$

$$\text{Time period}=\frac{2\pi}{\sqrt{\lambda}}$$

using value of $\lambda$ from ⑨

$$T=2\pi\sqrt{\frac{u^2-v^2}{\beta^2-\alpha^2}}$$

## Amplitude #

from ③

$$\alpha=\lambda x_1$$

$$\Rightarrow x_1=\frac{\alpha}{\lambda}$$

putting in ④

$$u^2=\lambda\left(a^2-\frac{\alpha^2}{\lambda^2}\right)$$

$$=\lambda a^2-\frac{\alpha^2}{\lambda}$$

$$\lambda a^2=u^2+\frac{\alpha^2}{\lambda}$$

$$\Rightarrow a^2=\frac{u^2}{\lambda}+\frac{\alpha^2}{\lambda^2}$$

using value of $\lambda = \frac{\beta^2-\alpha^2}{u^2-v^2}$ from ⑨

$$a^2 = u^2\left(\frac{u^2-v^2}{\beta^2-\alpha^2}\right) + \alpha^2\left(\frac{u^2-v^2}{\beta^2-\alpha^2}\right)^2$$

$$= \frac{u^2-v^2}{\beta^2-\alpha^2}\left[u^2 + \alpha^2\left(\frac{u^2-v^2}{\beta^2-\alpha^2}\right)\right]$$

$$= \frac{u^2-v^2}{\beta^2-\alpha^2}\left[\frac{u^2(\beta^2-\alpha^2)+\alpha^2(u^2-v^2)}{\beta^2-\alpha^2}\right]$$

$$= \frac{u^2-v^2}{\beta^2-\alpha^2}\left[\frac{u^2\beta^2 - \cancel{u^2\alpha^2} + \cancel{\alpha^2u^2} - \alpha^2v^2}{\beta^2-\alpha^2}\right]$$

$$= \frac{(u^2-v^2)(u^2\beta^2-\alpha^2v^2)}{\beta^2-\alpha^2}$$

$$a = \sqrt{\frac{(u^2-v^2)(u^2\beta^2-\alpha^2v^2)}{\beta^2-\alpha^2}}$$

$$\text{Amplitude} = \sqrt{\frac{(u^2-v^2)(u^2\beta^2-\alpha^2v^2)}{\beta^2-\alpha^2}}$$

**Q:12** # At the end of three consecutive seconds the distances of a point moving in S.H.M from its mean position are $x_1, x_2, x_3$ respectively. Show that the time of a complete oscillation is $\dfrac{2\pi}{\cos^{-1}\left(\frac{x_1+x_2}{2x_2}\right)}$

**Sol** # Suppose that when $x=a$, $t=0$, then distance $x$ at any time $t$ is given by

$$x = a\cos\sqrt{\lambda}\,t \longrightarrow ①$$

Let $x_1, x_2, x_3$ be the distances from the mean position after time $t$, $t+1$ and $t+2$ seconds

Then equation ① gives

$$x_1 = a\cos\sqrt{\lambda}\,t \longrightarrow ②$$
$$x_2 = a\cos\sqrt{\lambda}(t+1) \longrightarrow ③$$
$$x_3 = a\cos\sqrt{\lambda}\,(t+2) \longrightarrow ④$$

Adding ② & ④, we have

$$x_1 + x_3 = a\left[\cos\sqrt{\lambda}\,t + \cos\sqrt{\lambda}(t+2)\right]$$
$$= a\cdot 2\cos\left(\frac{\sqrt{\lambda}t+\sqrt{\lambda}t+2\sqrt{\lambda}}{2}\right)\cos\left(\frac{\sqrt{\lambda}t+2\sqrt{\lambda}-\sqrt{\lambda}t}{2}\right)$$
$$= 2a\cos\left(\frac{2\sqrt{\lambda}t+2\sqrt{\lambda}}{2}\right)\cos\left(\frac{2\sqrt{\lambda}}{2}\right)$$
$$= 2a\cos(\sqrt{\lambda}t+\sqrt{\lambda})\cos\sqrt{\lambda}$$
$$= 2a\cos\sqrt{\lambda}(t+1)\cos\sqrt{\lambda}$$
$$= 2x_2\cos\sqrt{\lambda}$$
$$\Rightarrow \sqrt{\lambda} = \cos^{-1}\left(\frac{x_1+x_2}{2x_2}\right)$$

Hence the time period of complete oscillation is

$$T = \frac{2\pi}{\sqrt{\lambda}} = \frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_2}{2x_2}\right)}$$

---

Q:13 # A point moves in a straight line with SHM. It has velocity $v_1$ when its distance from the centre is $x_1$ and has velocity $v_2$ when its distance from the centre is $x_2$. Show that time period is

$$2\pi\sqrt{\frac{x_1^2+x_2^2}{v_2^2-v_1^2}}$$

**Sol#** Since the motion of the particle is SHM, the velocity at any distance $x$ from the centre of motion is given by

$$v = \sqrt{\lambda}\sqrt{a^2-x^2}$$

squaring

$$v^2 = \lambda(a^2-x^2) \longrightarrow ①$$

at $x = x_1$ $v = v_1$

$$\Rightarrow \quad v_1^2 = \lambda(a^2-x_1^2) \longrightarrow ②$$

at $x = x_2$ $v = v_2$ and from ①

$$v_2^2 = \lambda(a^2-x_2^2) \longrightarrow ③$$

Subtracting ③ from ②

$$v_2^2 - v_1^2 = \lambda(x_2^2 - x_1^2)$$

$$\Rightarrow \lambda = \frac{v_2^2-v_1^2}{x_2^2-x_1^2}$$

$$\text{Time period} = \frac{2\pi}{\sqrt{\lambda}}$$

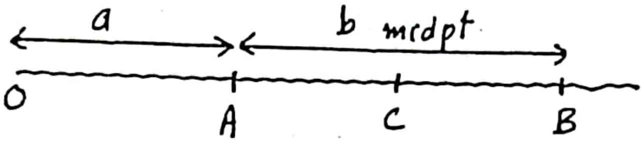
$$= \frac{2\pi}{\sqrt{\dfrac{v_2^2-v_1^2}{x_2^2-x_1^2}}} = 2\pi\sqrt{\frac{x_2^2-x_1^2}{v_2^2-v_1^2}}$$

---

**Q:14#** A particle moves in a straight line OAB with SHM. It is at rest at A and B and $OA = a$ and $OB = b$, its velocity is $v$ when it is at the middle point of AB. Show that the period is $\frac{\pi(b-a)}{v}$

**Sol#** Since the velocity is zero at the extreme positions, therefore A & B are the extreme positions of the S.H.M and middle point of AB will be the centre of SHM.

Let the points A and B be taken in the +ve directions of the x-axis with $b > a$



Then $AB = b - a$

Amplitude $= \frac{1}{2} AB = \frac{b-a}{2}$

Since the velocity at mean position which is mid point of AB is max and is given by

$$v_{max} = \sqrt{\lambda}\, a = \sqrt{\lambda}\,(\text{amplitude})$$

$$\Rightarrow \quad v = \sqrt{\lambda}\left(\frac{b-a}{2}\right)$$

$$\Rightarrow \quad \sqrt{\lambda} = \frac{2v}{b-a}$$

Hence we have

$$\text{Time period} = T = \frac{2\pi}{\sqrt{\lambda}} = \frac{2\pi}{\frac{2v}{b-a}}$$

$$= \frac{\pi(b-a)}{v} \quad \text{as required.}$$

**Q:15#** If a particle excecuting S.H.M starts from a point distant (a, from the centre with a speed $u$. Show that its distance from the centre after $t$ seconds is

$$\sqrt{a^2 + \frac{u^2}{\lambda}}\, \cos\left[\sqrt{\lambda}\, t - \tan^{-1}\frac{u}{a\sqrt{\lambda}}\right]$$

**Sol#** Since the particle doesnot start from mean position and xtreme position, therefore there will be some initial phase of motion of particle. Let $\varphi$ be the initial phase of the motion. Then general form of displacement is given by

$$x = A\cos(\sqrt{\lambda}\, t + \varphi) \longrightarrow ①$$

When $t=0$ $x=a$ and ① becomes

$$a = A\cos\varphi \longrightarrow ②$$

Diff ① w.r.t. $t$

$$v = \dot{x} = -A\sqrt{\lambda}\,\sin(\sqrt{\lambda}t+\varphi)$$

when $t=0$ ~~$x=a$~~ $v=u$

$$\Rightarrow u = -A\sqrt{\lambda}\sin\varphi$$

$$\Rightarrow -\frac{u}{\sqrt{\lambda}} = A\sin\varphi \longrightarrow ③$$

squaring and adding ② & ③

$$a^2+\frac{u^2}{\lambda} = A^2$$

$$\Rightarrow A = \sqrt{a^2+\frac{u^2}{\lambda}} \longrightarrow ④$$

Dividing ③ by ②

$$-\frac{u}{a\sqrt{\lambda}} = \tan\varphi = \frac{a}{-\frac{u}{\sqrt{\lambda}}} = -\frac{a\sqrt{\lambda}}{u}$$

$$\tan^{-1}\left(-\frac{u}{a\sqrt{\lambda}}\right) = \varphi = \tan^{-1}\left(-\frac{a\sqrt{\lambda}}{u}\right) \longrightarrow ⑤$$

using values of $A$ & $\varphi$ in ①

Required distance after $t$ seconds is

$$x = \sqrt{a^2+\frac{u^2}{\lambda}}\,\cos\left[\sqrt{\lambda}t - \tan^{-1}\left(\frac{u}{a\sqrt{\lambda}}\right)\right]$$

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Q:16 # A point performing SHM of period $T$ about centre $O$ and it passes through the position $P$ ($OP=b$) with velocity $v$ in the direction of $OP$. Prove that the time that elapses before it returns to $P$ is

$$\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$$

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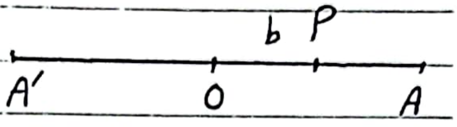
**Sol#** Let time be measured when the particle is at the extreme position A. Then equation of S.H.M will be



$$x = a\cos\sqrt{\lambda}\,t \longrightarrow ①$$

At point P $x = b$, $t = t_1$ say (time for motion from A to P)

$$\Rightarrow b = a\cos\sqrt{\lambda}\,t_1 \longrightarrow ②$$

Since

$$vel = \sqrt{\lambda}\sqrt{a^2 - x^2}$$

$$\Rightarrow (vel)^2 = \lambda(a^2 - x^2)$$

at point P $vel = v$ $x = b$

$$v^2 = \lambda(a^2 - b^2)$$

$$= \lambda(a^2 - a^2\cos^2\sqrt{\lambda}\,t_1) \quad \text{using ②}$$

$$v^2 = \lambda(a^2\sin^2\sqrt{\lambda}\,t_1) \longrightarrow ③$$

Also $T = \frac{2\pi}{\sqrt{\lambda}}$

$$\Rightarrow \sqrt{\lambda} = \frac{2\pi}{T} \longrightarrow ④$$

Taking square root of ③

$$v = \sqrt{\lambda}\,a\sin\sqrt{\lambda}\,t_1 \longrightarrow ⑤$$

Dividing ⑤ by ②

$$\frac{v}{b} = \sqrt{\lambda}\tan\sqrt{\lambda}\,t_1$$

$$\Rightarrow \tan\sqrt{\lambda}\,t_1 = \frac{v}{b\sqrt{\lambda}}$$

using $\sqrt{\lambda}$ from ④

$$\tan\sqrt{\lambda}\,t_1 = \frac{v}{b\cdot\frac{2\pi}{T}} = \frac{vT}{2\pi b}$$

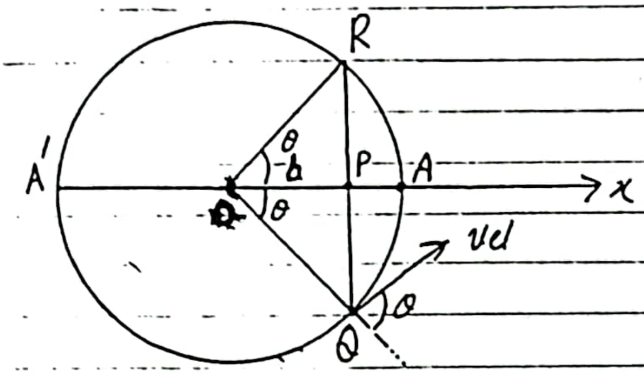
$$\sqrt{\lambda}\,t_1 = \tan^{-1}\left(\frac{vT}{2\pi b}\right)$$

$$\frac{2\pi}{T}t_1 = \tan^{-1}\left(\frac{vT}{2\pi b}\right)$$

$$2t_1 = \frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$$

which is required time from A to P and then from P to A.

## Method II (by uniform circular Motion)



Let (a) be the amplitude of SHM and A, A' be extreme positions. Consider the uniform motion of the projection of particle on an associated circle of radius a. Suppose when particle performing SHM is at point P on diameter, its projection on circle is at point Q and when particle in SHM returns back to P from A, its projection on circle is at point R.

Time taken by particle to return back to P from A is the same as the time taken by its projection on circle to cover arc $\overset{\frown}{QAR}$ with uniform angular velocity ω.

Let $\angle POQ = \theta$, then $\angle ROP = \theta$

( ∵ ΔOQR is an isoceles and OP is ⊥ar to QR)

Now

$$T = \frac{2\pi}{\omega}$$

$$\Rightarrow \omega = \frac{2\pi}{T} \longrightarrow ①$$

linear velocity along circle $= a\omega$

when particle in SHM is at P its velocity $v$ is

$$v = a\omega \sin\theta$$

putting $\omega = \frac{2\pi}{T}$

$$v = \frac{2a\pi}{T} \sin\theta \longrightarrow ②$$

Also from $\triangle OPQ$

$$\frac{b}{a} = \cos\theta$$

$$b = a\cos\theta \longrightarrow ③$$

Dividing ② by ③

$$\frac{v}{b} = \frac{2\pi}{T}\tan\theta$$

$$\Rightarrow \tan\theta = \frac{vT}{2b\pi}$$

$$\theta = \tan^{-1}\left(\frac{vT}{2b\pi}\right) \longrightarrow ④$$

Let t be the time to covere arc $\widehat{QAR}$. Then

$$t = \frac{2\theta}{\omega}$$

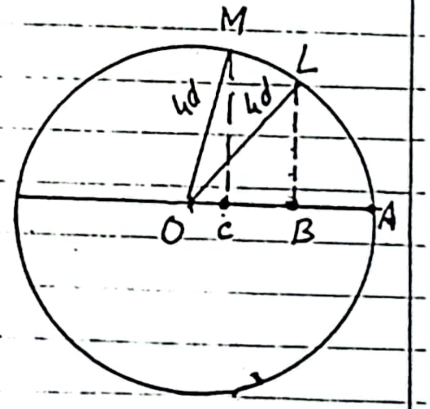
$$= \frac{2\theta}{\frac{2\pi}{T}}$$

$$t = \frac{T}{\pi}\theta = \frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2b\pi}\right)$$

as required

---

**Q:17#** A particle is performing linear SHM about a centre O and with amplitude $OA = 4d$. B and C are two points on the path such that $OB = 2d$, $OC = d$. Find the time taken to travel (a) from A to B (b) from A to C if one oscillation is completed in $\pi$ seconds.



**Sol#** Project B and C onto the associated circle at L and M. Let the constant angular velocity around the associated circle be ω rad/sec

It is given that $T = \pi$

$\Rightarrow \quad T = \frac{2\pi}{\omega}$

$\Rightarrow \quad \omega = \frac{2\pi}{\pi} = 2$

Now $OL$ = radius = amplitude $= 4d$

$OB = 2d \qquad OC = d$

$\cos\angle LOA = \frac{2d}{4d} = \frac{1}{2}.$

$\Rightarrow \angle LOA = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$

$\cos\angle MOA = \frac{d}{4d} = \frac{1}{4}.$

$\Rightarrow \angle MOA = \cos^{-1}(\frac{1}{4}) \simeq 1.32^\circ$

## Time from A to B #

Time from A to B = time from A to L with constant angular velocity $\omega$

$= \frac{\text{angle}}{\text{angular vel}}$

$= \frac{\frac{\pi}{3}}{2} = \frac{\pi}{6}$ seconds

## Time from A to C #

Time from A to C = time from A to M

$= \frac{\text{angle}}{\text{angular vel}} = \frac{1.32^\circ}{2}$

$\simeq .66$ seconds

## Alternate Method #

we have $\omega = 2$ rad/sec

Since the time is measured from A.

$x = a\cos\omega t \rightarrow ① \qquad \omega = \sqrt{\lambda}$

At point B #

$x = 2d$. Let $t_1$ be time from A to B

Then $2d = 4d\cos 2t_1$

$$2t_1 = \cos^{-1}(\tfrac{1}{2}) = \pi/3$$

$$t_1 = \pi/6 \text{ seconds}$$

**At Point C**

$x = d$ let time from A to C be $t_2$

$$d = 4d\cos 2t_2$$

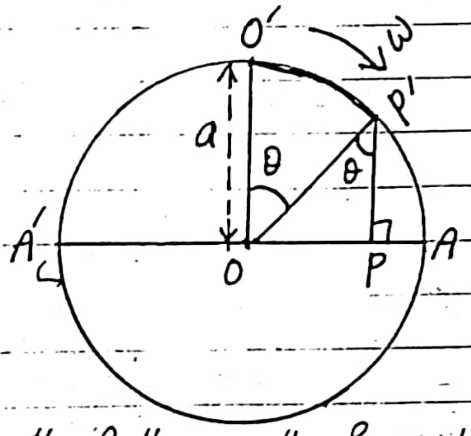
$$2t_2 = \cos^{-1}(\tfrac{1}{4}) = 1.32^\circ$$

$$t_2 = \frac{1.32^\circ}{2} = .66 \text{ seconds.}$$

---

**Q:18 #** A particle travelling with linear SHM of period T starts from centre O of its path which of length 2a. The particle travels for a time t and is then at point P. Find the distance OP if (a) $t = \frac{1}{6}T$ (b) $t = \frac{1}{3}T$

**Sol #**



Since length of the path of particle and it starts from centre O, therefore amplitude = a = radius of the associated circle.

As p travels along AA' with SHM its projection p' on the associated circle travels with constant angular velocity

$$T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T} \rightarrow ①$$

The time (t) taken to travel from O to P with SHM is equal to the time taken to traverse arc O'P' with constant angular velocity i.e

$$t = \frac{\theta}{\omega} \Rightarrow \theta = \omega t \rightarrow ②$$

But $OP = a\sin\theta = a\sin\omega t$

$$op = a\sin\frac{2\pi}{T}t \longrightarrow ③$$

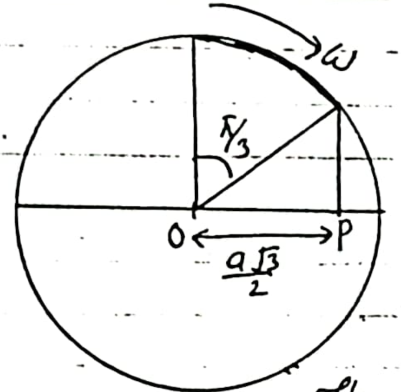
(a) If $t = \frac{1}{6}T$, then from ③

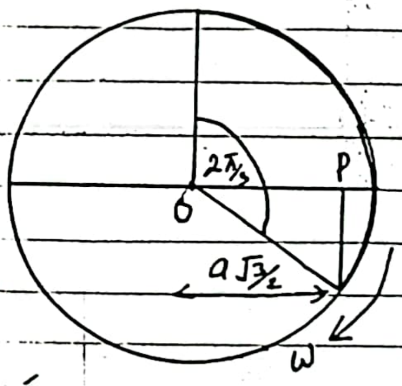
$$op = a\sin\pi/3 = a\frac{\sqrt{3}}{2}$$

(b) If $t = \frac{1}{3}T$, then from ③

$$op = a\sin\frac{2\pi}{3} = \frac{a\sqrt{3}}{2}$$

Therefore P is at the same distance from O when $t = \frac{1}{6}T$ and $t = \frac{1}{3}T$ but is travelling in the 1st case away from O and in the 2nd case towards O





The End

تمت بالخیر

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